Finer properties of Harmonic measure

Ilia Binder, Adi Glücksam

1 Introduction

Let $\Omega \subset \mathbb{C}$ be a simply connected domain, $z_0 \in \Omega$, and let $\phi : \mathbb{D} \to \Omega$ be a conformal mapping so that $\phi(0) = z_0$. The *harmonic measure* of Ω at z_0, $\omega : \mathcal{B}(\partial\Omega) \to [0,1]$, can be defined by

$$\omega(E) := \omega(z_0, E; \Omega) = \lambda_1\left(\phi_*^{-1}(E)\right),$$

where ϕ_* denotes the boundary extension of ϕ and λ_1 denotes the one dimensional Hausdorff measure on the torus, normalized to that $\lambda_1(\mathbb{T}) = 1$ (see [29]).

In his celebrated paper [43], Makarov established that

$$\dim \omega = \inf \dim_H \{K : \omega(K) = 1\} = 1 \text{ and } \underline{\dim}\, \omega = \inf \dim_H \{K : \omega(K) > 0\} = 1.$$

In this paper, we aim to explore the multifractal analysis of harmonic measure and rotation in arbitrary simply connected domains. For a detailed discussion of the multifractal analysis, we refer the reader to [26]. This work continues the analysis initiated in [45]. Our main goal is to extend the results of [45] to the rotational case and provide proofs of some "folklore" results. In the process, we encountered a few minor surprises requiring new techniques to handle, as well as a major new phenomenon (Theorem 2.2 and Theorem 2.3). This new phenomenon necessitates using a new technique for "dual" fractal approximation (Section 5).

Let us now turn to a more careful informal description of the results. As before, let Ω be a simply connected domain. The dimension mixed spectrum is a continuum of parameters, $f_\Omega(\alpha, \gamma)$, defined for all $\alpha > 0, \gamma \in \mathbb{R}$, which characterizes the "harmonic" dimension (i.e., dimension with respect to harmonic measure) of the boundary set with a prescribed speed of rotation of Green lines and prescribed local dimension of the harmonic measure. In simple terms, $f(\alpha, \gamma)$ is the dimension of the set of points $\left\{x \in \partial\Omega : \lim_{\delta\to 0} \frac{\log(\omega(B(x,\delta)))}{\log(\delta)} = \alpha,\ \lim_{\delta\to 0} \frac{\log(rho(x,\delta))}{\log(\delta)} = \gamma\right\}$, where *rho* is the rotation defined in section 2. The exact definition has different versions, involving lim sup or lim inf and Hausdorff or Minkowski dimensions. These versions are often quite different, even for polygonal domains. However, they have the same universal bounds. We refer the reader to Section 2 for more in-depth discussion.

Dimension mixed spectra provide a wealth of information about the geometry of the boundary of a planar domain.

In particular, the spectra describe all possible local dimensions and rotation speeds, as well as the prevalent rotation speed in the sense of Hausdorff measures of different dimensions.

The distortion mixed spectrum is the conformal map counterpart of the dimension spectrum. It is denoted by $d_\Omega(a,b)$, where $a > 0$ and $b \in \mathbb{R}$. In simple terms, it is the dimension of the set $\zeta \in S^1$ for which $|\phi'(r\zeta)|$ grows like $\left(\frac{1}{1-r}\right)^a$, and $\exp(\arg \phi'(r\zeta))$ grows like $\left(\frac{1}{1-r}\right)^b$. Again, there are different variants of the definition involving Minkowski and Hausdorff dimensions. See Section 2 for precise definitions.

Since the function $\log \phi'(\zeta)$ is a Bloch function, one can construct a Bloch martingale associated with it and then apply the theory of large deviations (for the description of Bloch martingales, we referthe reader to [44], and for the discussion of large deviations, we refer to [25]). Along these lines, the entropy function for the distortion mixed spectrum is the integral mixed spectrum:

$$m_\Omega(z) = \limsup_{r\to 1^-} \frac{\log \int_{rS^1} |\phi'^z(\zeta)||d|\zeta|}{\log \frac{1}{1-r}} .$$

Note that in the case of real z, this object is classical and has been extensively studied (for example, [45] and [48]). We need to introduce the complex exponent here to reflect the properties of the rotation. Most of the classical results can be easily carried out to the case of complex exponent. The values of the integral mixed spectrum for real z correspond to the behavior of harmonic measure, and the purely imaginary exponents z can be considered "rotational".

It is worth noting that, due to the entropy relation mentioned earlier, the integral mixed spectrum and the distortion mixed spectrum (to be precise, the Minkowski distortion spectrum) are related by a Legendre-type transform:

$$m_\Omega(z) = \sup_{a,b} \left(d_\Omega(a,b) + a\mathbb{R}e\,[z] + b\mathbb{I}m\,[z] - 1\right) \tag{1}$$

$$d_\Omega(a,b) = \inf_{z} \left(m_\Omega(z) - a\mathbb{R}e\,[z] - b\mathbb{I}m\,[z] + 1\right) \tag{2}$$

As mentioned above, harmonic measure of a simply connected domain can be defined as the pushforward of the normalized linear measure on the circle under conformal map (see [29] for more details), it is natural to relate the boundary behaviour of the derivative of conformal map and the local dimension of harmonic measure.

Intuitively, one expects the relationship

$$d_\Omega(a,b) = (1-a) f_\Omega\left(\frac{1}{1-a}, \frac{-b}{1-a}\right).$$

This relation indeed holds for domains with quasi-circular boundaries (see Theorem 2.1 for partial justification).

It is well-known that for all versions of the spectra there are examples (see [45]) with

$$d_\Omega(a,b) > (1-a) f_\Omega\left(\frac{1}{1-a}, \frac{-b}{1-a}\right).$$

In this paper, we provide examples of domains where opposite inequality holds (Theorem 2.2 and Theorem 2.3). This phenomenon was not expected by experts in the field.

The multifractal spectra, which were defined earlier, have additional properties for domains with boundaries that are invariant under a hyperbolic dynamical system, known as the "Jordan repellers". Examples of such repellers are the basin of attraction to infinity of a hyperbolic polynomial or a snowflake domain (also called Carleson fractal). The multifractal spectra for Jordan repellers are thermodynamic objects and can be defined in terms of the pressures of some potentials related to the dynamics on the boundaries and the dimensions of the corresponding Gibbs measures. This allows us to apply the techniques of thermodynamic formalism to understand the relations between the mixed spectra and their behavior. The Minkowski and Hausdorff versions of the spectra coincide, and all the multifractal spectra for Jordan repellers are real analytic, exist as limits, and related by Legendre-type transform. For more details, please refer to [45].

In Section 5, we provide a new proof of the "Fractal Approximation phenomenon". We show that the universal bounds for distortion and dimension spectrum for bounded domains can be obtained by considering only Jordan repellers. Since, for these domains, all versions of the spectra agree and the dimension and distortion spectra are related by Legendre-type transform, the same relation holds for the universal bounds on the spectra. We need to study the fractal approximation for both dimension and distortion spectra because of the phenomenon established in Theorem 2.3. We would also like to point out that carrying out the approximation required a refinement of the classical lemma due to L. Carleson (Lemma 5.14).

2 Background, definitions, and results

For every $a, b \in \mathbb{R}$ we define the *Minkowski distortion mixed spectrum* of Ω as

$$d_\Omega(a,b) := \lim_{\substack{a'\to a \\ b'\to b}} \limsup_{r\to 1^-} \frac{\log\left(\lambda_1\left(L_{a',b'}(r)\right)\right)}{\log\left(\frac{1}{1-r}\right)} + 1,$$

where

$$L_{a',b'}(r) := \left\{\zeta \in \partial D, \frac{\log|\phi'(r\zeta)|}{a'} > \log\left(\frac{1}{1-r}\right) \text{ and } \frac{arg\left(\phi'(r\zeta)\right)}{b'} > \log\left(\frac{1}{1-r}\right)\right\},$$

arg being the branch of the argument of ϕ' with $Arg(\phi'(0)) \in (-\pi, \pi]$.

We define the *rotation of a domain, Ω, around a boundary point z* by

$$rot(z,\delta) := \exp\left(\inf_{y\in\partial\Omega_\delta\cap\partial B(z,\delta)} arg_{[z]}(y-z)\right),$$

where Ω_δ is the connected component of the set $\{y \in \Omega,\ |y-z| > \delta\}$ containing z_0, and the argument $arg_{[z]}$ is a branch of the argument satisfying that $arg_{[z]}(z_0 - z) \in (-\pi, \pi]$.

For every $\alpha, \gamma \in \mathbb{R}$ we define the *Minkowski dimension mixed spectrum* of Ω as

$$f_\Omega(\alpha,\gamma) = \lim_{\eta\to 0} \limsup_{\delta\to 0} \frac{\log N(\delta,\alpha,\gamma,\eta)}{\log\left(\frac{1}{\delta}\right)},$$

where $N(\delta,\alpha,\gamma,\eta)$ is the maximal number of disjoint disks $\{B(z_j,\delta)\}$ satisfying that

1. $z_j \in \partial\Omega$.

2. $\forall j \neq k,\ B(z_j,\delta) \cap B(z_k,\delta) = \emptyset$.

3. $\omega(B(z_j,\delta)) \in (\delta^{\alpha+\eta}, \delta^{\alpha-\eta})$.

4. $rot(z_j,\delta) \in (\delta^{\gamma+\eta}, \delta^{\gamma-\eta})$.

While attempting the proof the authors encountered a problem- it is possible that every curve in the disk B_k either carries a large portion of the harmonic measure or has a large enough diameter, but not both, and extending the curve would either increase the harmonic measure by too much or make it too long. Such disks should be counted in fact in a different smaller scale. To overcome this issue it en enough to assume that the domain is a quasi-disk, which is the primary case, as we will later see (see Theorem 2.4).

In this note we will prove a theorem, originally proven by Makarov, showing that the Minkowski dimension mixed spectrum is dominated by the Minkowski distortion mixed spectrum (with the correct parameter) if the harmonic measure is doubling, and show it is not correct without some additional assumptions on the domain. In fact, we generate an example showing that the local Hausdorff dimension is not always dominated by the Minkowski dimension mixed spectrum. However, we will show that the universal counterparts do satisfy this relation.

Theorem 2.1 *Let $\Omega \subset \mathbb{C}$ be a quasi-disk. Then*

$$d_\Omega(a,b) \geq (1-a)\, f_\Omega\left(\frac{1}{1-a}, \frac{-b}{1-a}\right).$$

The original version of this theorem was proved by Makarov in [45]. They overlooked the case where we 'look at the wrong scale', i.e. when we work with disks where the main arc, which carries most of the harmonic measure, has a very small diameter. An extension to Makarov's version of this theorem was proven by Binder in [14].

Theorem 2.2 *For every $a \in \left(0, \frac{1}{3}\right)$ there exists a domain $\Omega \subset \mathbb{C}$ whose boundary is a Jordan curve and has only one cusp, satisfying that*

$$d_\Omega(a) < (1-a)\, f_\Omega\left(\frac{1}{1-a}\right).$$

Lastly, we define the function

$$\tilde{f}_\Omega(\alpha) := \lim_{\eta\to 0^+} dim\left(\left\{z, \exists\{\delta_k\} \searrow 0, \delta_k^{\alpha+\eta} \leq \omega(B(z,\delta_k)) \leq \delta_k^{\alpha-\eta}\right\}\right)$$

In fact, it is not even the case that in general

$$d_\Omega\left(1-\frac{1}{\alpha}\right) \geq \frac{1}{\alpha}\tilde{f}_\Omega(\alpha)$$

as the following example shows:

Theorem 2.3 *For every $\alpha > 1$ there exists a domain $\Omega \subset \mathbb{C}$ such that*

$$d_\Omega\left(1-\frac{1}{\alpha}\right) < \frac{1}{\alpha}\tilde{f}_\Omega(\alpha).$$

Finally, we will show that while the inequalities presented here are not true for every domain, they do hold for their universal counterpart.

Theorem 2.4 *1. $F(\alpha) := \sup_{\substack{\Omega \\ s.c}} f_\Omega(\alpha) = F^+(\alpha) = \sup_{F\ IFS} f^+_{\Omega_F}(\alpha)$ for all $\alpha > 0$.*

2. $D(a) := \sup_{\substack{\Omega \\ s.c}} d_\Omega(a) = \sup_{F\ IFS} d_{\Omega_F}(a)$ for all $a > 0$.

In particular,

$$D\left(1-\frac{1}{\alpha}\right) = \frac{1}{\alpha}F(\alpha).$$

3 The proof of Theorem 2.1

3.1 Auxiliary Results for the Proof

In this section we present all the required auxiliary definitions and results needed to prove Theorem 2.1.

3.1.1 Counting curves and distortion spectrum

The first subsection will relate the Miskowski distortion spectrum with a collection of curves.

Definition 3.1 *For every $r \in (0,1)$ and $a > 0$ fixed we define by $\Gamma(a,r)$ to be the maximal collection of disjoint curves from the collection*

$$\left\{\gamma \subset \partial\Omega, \exists A \subset \mathbb{T}, \lambda_1(A) = (1-r), \phi(A) = \gamma, \text{ and } diam(\gamma) \geq (1-r)^{1-a}\right\}, \text{ if } a > 0$$

We then define the **Minkowski curve-distortion spectrum** *by*

$$d^{curve}(a) = \limsup_{a' \searrow a} \limsup_{r \nearrow 1} \frac{\log(\#\Gamma(a', r))}{\log\left(\frac{1}{1-r}\right)}.$$

In a sense, if $\alpha = \frac{1}{1-a}$ and $1-r = \varepsilon^\alpha$ then these curves satisfy that the harmonic measure of each curve is ε^α and its diameter is bounded from below (above) by ε if $\alpha > 1$ (if $\alpha < 1$).

The first Lemma in this subsection shows that there is some correspondence between the Minkowski curve-distortion spectrum, d^{curve}, and the Minkowski distortion spectrum d, and between the Minkowski curve distortion spectrum and the universal Minkowski dimension spectrum, F.

Lemma 3.2 *1. If $a > 0$, then for every simply connected domain, Ω, $d_\Omega(a) \le d_\Omega^{curve}(a)$.*

2. If $a < 0$, then for every simply connected domain, Ω, $d_\Omega(a) \le (1-a)\, F\left(\frac{1}{1-a}\right)$.

3. If $a > 0$ and Ω is a quasi-disk, then $d_\Omega(a) = d_\Omega^{curve}(a)$.

Proof. The essence of the proof lies in the following observation:

Observation 3.3 *Let $\phi : \mathbb{D} \to \mathbb{C}$ be a conformal map and fix $I \subset \partial\mathbb{D}$ some arc with $\lambda_1(I) < \frac{1}{2}$. Denote by ζ_I the centre of the arc, I, and let $z_I = \zeta_I\,(1-\lambda_1(I))$. Then*

1.

$$dist(\phi(z_I), \phi(I)) \lesssim_\Omega diam(\phi(I)).$$

2. If Ω is a quasi-disk, and K is the smallest dilatation of the quasi-conformal extension of ϕ to the Riemann sphere. Then for every z,

$$C(K)\,(1-|z|)\,|\phi'(z)| \le diam(\phi(I)) \le C(K)^{-1}\,(1-|z|)\,|\phi'(z)|\,.$$

The proof is given in [29] exercises 8 on p. 153 for the first part, and on p. 216 for the second part.

The proof of 1: Let $a' > a$, fix r close enough to 1, and let

$$L_{a'}(r) = \left\{\zeta \in \partial\mathbb{D}, \log|\varphi'(r\zeta)| > a' \log\left(\frac{1}{1-r}\right)\right\}.$$

Let $\Gamma = \{A_j\}_{j=1}^M$ denote the minimal collection of disjoint arcs in $\partial\mathbb{D}$ satisfying that $diam(\gamma) = 1-r$ while $\bigcup_{j=1}^M 2A_j \supset L_{a'}(r)$. For every j there exists $\zeta' \in 2A_j \cap L_{a'}(r) \neq \emptyset$, as $\{2A_j\}$ forms a cover for $L_{a'}(r)$. Let a_j denote the centre of the arc A_j. The function ϕ is conformal making $\log\phi'$ a Bloch function, therefore there exists a uniform constant C_Ω so that for all $\zeta \in 2A_j$

$$\left|\log|\phi'(r\cdot\zeta)| - \log|\phi'(r\cdot a_j)|\right| \le C_\Omega,$$

and for every $\zeta \in 2A_j \cap L_{a'}(r)$

$$\log|\phi'(r\cdot a_j)| \ge \log|\phi'(r\cdot\zeta)| - C_\Omega \ge a'\log\left(\frac{1}{1-r}\right) - C_\Omega \ge \left(a' - \frac{C_\Omega}{\log\left(\frac{1}{1-r}\right)}\right)\log\left(\frac{1}{1-r}\right).$$

For every j let $\gamma_j := \phi(A_j)$, and note that, following Koebe's distortion theorem combined with the first part of Observation 3.3,

$$diam(\gamma_j) \gtrsim |\phi'(a_j)|\,(1-|a_j|) \ge \left(\frac{1}{1-r}\right)^{a' - \frac{C_\Omega}{\log\left(\frac{1}{1-r}\right)} - 1}$$

that is there exists another constant, which depends on Ω satisfying

$$diam(\gamma_j) \geq (1-r)^{1-a'+\frac{C'_\Omega}{\log\left(\frac{1}{1-r}\right)}}.$$

Define $a'' := a' - \frac{C'_\Omega}{\log\left(\frac{1}{1-r}\right)}$, then as long as r is close enough (depending on a') $a'' > a$ and

$$diam(\phi(A_j)) = diam(\gamma_j) \geq \left(\frac{1}{1-r}\right)^{a' - \frac{C_\Omega}{\log\left(\frac{1}{1-r}\right)} - 1} = (1-r)^{a''-1}.$$

In particular,

$$M = \text{number of curves } \gamma_j \leq \#\Gamma\left(a'', r\right),$$

implying that

$$\begin{aligned} d_\Omega\left(a\right) &= \limsup_{a' \searrow a} \limsup_{r \nearrow 1} \frac{\log\left(\lambda_1\left(L_{a'}(r)\right)\right)}{\log\left(\frac{1}{1-r}\right)} + 1 = \limsup_{a' \searrow a} \limsup_{r \nearrow 1} \frac{\log\left(2(1-r)\cdot M\right)}{\log\left(\frac{1}{1-r}\right)} + 1 \\ &\leq \limsup_{a' \searrow a} \limsup_{r \nearrow 1} \frac{\log\left(\#\Gamma\left(a'', r\right)\right)}{\log\left(\frac{1}{1-r}\right)} \leq d^{curve}(a), \end{aligned}$$

concluding the proof of 1.

The proof of 2: Note that for $a < 0$ the function $a \mapsto d_\Omega(a)$ is monotone increasing since if $a' < a$ then $|a'| > |a|$ and since $(1-r) < 1$ we get

$$|\phi'(r\zeta)| \leq (1-r)^{|a'|} < (1-r)^{|a|}.$$

We will prove the theorem for a such that for all $\eta > 0$ we have $d_\Omega(a-\eta) < d_\Omega(a)$. If this is not the case, there exists $a' < a$ for which it does hold, and as $\alpha(a') = \frac{1}{1-a'}$ is a monotone increasing function and f^+_Ω is monotone increasing, we will get that

$$F^+(\alpha(a)) \geq F^+(\alpha(a')) \geq d_\Omega(a') = d_\Omega(a).$$

We may therefore assume without loss of generality that for every $\eta > 0$ we have $d_\Omega(a) > d_\Omega(a-\eta)$. Fix $a' > a$ and $\delta_0 > 0$, and let $\eta := |a - a'|$ and $\varepsilon := \frac{1}{4}\left(d_\Omega(a') - d_\Omega(a' - 2\eta)\right) > 0$. There exists $r > r_0 := 1 - \delta_0^{\frac{1}{1-a'}}$ large enough so that

$$\frac{\log\left(\lambda_1(L_{a'}(r))\right)}{\log\left(\frac{1}{1-r}\right)} + 1 + \varepsilon > d_\Omega(a') > d_\Omega(a'-2\eta) > \frac{\log\left(\lambda_1(L_{a'-2\eta}(r))\right)}{\log\left(\frac{1}{1-r}\right)} + 1 - \varepsilon.$$

Note that while it is possible that $L_{a'-2\eta}(r) \ll (1-r)^{1-d_\Omega(a'-2\eta)}$ an inequality still holds as long as r is large enough.

We partition $\mathbb{T}$ into $n := \left\lceil \frac{1}{1-r} \right\rceil$ and let $I_j \subset \partial\mathbb{D}$ be the minimal collection of such arcs with $L_{a'}(r) \subset \uplus_j 2I_j$. In particular, for every j we have $2I_j \cap L_{a'}(r) \neq \emptyset$, implying that for every $\zeta \in I_j$

$$|\phi'(r\cdot\zeta)| \leq C\,(1-r)^{|a|'}$$

for some uniform constant C. Let J denote the collection of indices j so that for every $\zeta \in I_j$, $|\phi'(r\cdot\zeta)| > \frac{1}{C}(1-r)^{|a'|+2\eta}$. Since $a'-2\eta = a-\eta$ then $d_\Omega(a'-2\eta) < d_\Omega(a)$. Note that while it is possible that $L_{a'-2\eta}(r) \ll (1-r)^{1-d_\Omega(a'-2\eta)}$ an inequality still holds as long as r is large enough, and so

$$\begin{aligned}\#J &= \frac{\lambda_1\left(\biguplus_{j\in J} I_j\right)}{\frac{1}{n}} \geq n\cdot\lambda_1\left(L_{a'}(r)\setminus L_{a'-2\eta}(r)\right) \geq n\left((1-r)^{1+\varepsilon-d_\Omega(a')} - (1-r)^{1-\varepsilon-d_\Omega(a'-2\eta)}\right)\\ &= n(1-r)^{1+\varepsilon-d_\Omega(a')}\left(1-(1-r)^{d_\Omega(a')-d_\Omega(a'-2\eta)-2\varepsilon}\right) \geq \frac{1}{2}(1-r)^{\varepsilon-d_\Omega(a')},\end{aligned}$$

for r large enough, by the way ε was defined.

For every $j\in J$ denote by $z_j = (1-r)\zeta_j$ where ζ_j is the centre of I_j. Note that if $j\neq k$ then

$$|\phi(z_j)-\phi(z_k)| \gtrsim \rho(z_j,z_k)(1-r)\,|\phi'(z_j)| \gtrsim |z_j-z_k|\,(1-r)^{|a'|+2\eta} \geq (1-r)^{1+|a'|+2\eta}.$$

In particular, there exists $c>0$ uniform so that

$$B\left(\phi(z_j), c\,(1-r)^{1+|a'|+2\eta}\right)\cap B\left(\phi(z_k), c\,(1-r)^{1+|a'|+2\eta}\right) = \emptyset.$$

Let $\delta := c\,(1-r)^{1+|a'|+2\eta}$. Then the collection $\{B_j\} := \{B\left(\phi(z_j),\delta\right)\}$ is a collection of pairwise disjoint disks and for every j,

$$\omega(z_0, B_j; \phi(r\mathbb{D})) = \lambda_1(\phi^{-1}(B_j)) \gtrsim \lambda_1((1-r)^{1+2\eta}I_j) \sim (1-r)^{1+2\eta} \sim \delta^{\frac{1+2\eta}{1+|a'|+2\eta}}$$

since if $|z-z_j| < (1-r)^{1+2\eta}$, then

$$|\phi(z)-\phi(z_j)| = \int_z^{z_j} |\phi'(\zeta)||d\zeta| \lesssim (1-r)^{|a'|}\,(1-r)^{1+2\eta} = (1-r)^{1+|a'|+2\eta}.$$

In particular

$$N_{\phi(r\mathbb{D})}\left(\frac{1+2\eta}{1+|a'|+2\eta}, \delta, \eta\right) \geq \#J \geq \frac{1}{2}(1-r)^{\varepsilon-d_\Omega(a')}.$$

Taking first $r\nearrow 1$ and then $a'\searrow a$ we see that

$$(1-a)\sup_{\Omega\text{ s.c}} f_\Omega^+\left(\frac{1}{1-a}\right) \geq (1-a)\sup f_{\phi(r\mathbb{D})}^+\left(\frac{1}{1-a}\right) \geq d_\Omega(a).$$

Finally, since this is true for all simply connected Ω, this definitely holds for the supremum, $D(a)$.

The proof of 3: In light of 1, we only need to show that $d^{curve}(a) \leq d(a)$. In fact, all we need to show is that for every $\gamma\in\Gamma(a',r)$ there exists z_γ satisfying $|z_\gamma| = r$ and $|\phi'(z_\gamma)| \geq (1-r)^{1-a''}$ for some $a'' > a$. However, in light of the second part of Observation 3.3, taking $z_\gamma := r\cdot\zeta_\gamma$ for some $\zeta_\gamma\in A_\gamma$, and repeating the same argument as done in the proof of 1 concludes the proof. The only thing one needs to note is that

$$|\zeta_\gamma - \zeta_{\gamma'}| \geq (1-r)\,,$$

as they both sit in the centre of A_γ and $A_{\gamma'}$. □

3.1.2 Rotation

The rest of the lemmas in this section will reveal intriguing properties of the rotation. We will use the notation presented in the section 2. The first Lemma shows that one can estimate the rotation using integration over curves in Ω:

Lemma 3.4 *Let $w \in \partial\Omega$ and $\delta > 0$. For every curve $\gamma \subset \Omega$ connecting $y \in \partial B(w,\delta) \cap \Omega_\delta$ with z_0, we have*

$$\left| \mathbb{I}m \left[\int_\gamma \frac{1}{\xi - w} d\xi \right] - \log\left(rot(w,\delta)\right) \right| \leq 3\pi.$$

Proof. Let $\gamma_0 \subset \Omega$ be a curve connecting z_0 with w. Denote by γ_δ the connected component of $\gamma_0 \setminus B(w,\delta)$ which contains z_0, and let y_δ be the point where γ_δ ends. Then

$$\begin{aligned} 0 &= \int_{\gamma_0 - \gamma_0} \frac{1}{\xi - w} d\xi = \int_{\gamma_0} \frac{1}{\xi - w} d\xi - \left(\int_{\gamma_\delta} \frac{1}{\xi - w} d\xi + \int_{\gamma_0 \setminus \gamma_\delta} \frac{1}{\xi - w} d\xi \right) \\ \Longrightarrow 0 &= \mathbb{I}m \left[\int_{\gamma_0} \frac{1}{\xi - w} d\xi \right] - \left(\mathbb{I}m \left[\int_{\gamma_\delta} \frac{1}{\xi - w} d\xi \right] + \mathbb{I}m \left[\int_{\gamma_0 \setminus \gamma_\delta} \frac{1}{\xi - w} d\xi \right] \right) \\ &= arg_{[w]}(z_0 - w) - \mathbb{I}m \left[\int_{\gamma_\delta} \frac{1}{\xi - w} d\xi \right] - arg_{[w]}(y_\delta - w). \end{aligned}$$

Since the branch of the argument is chosen so that $arg_{[w]}(z_0 - w) \in (-\pi, \pi]$ we get that

$$\mathbb{I}m \left[\int_{\gamma_\delta} \frac{1}{\xi - w} d\xi \right] - \pi \leq arg_{[w]}(y_\delta - w) \leq \mathbb{I}m \left[\int_{\gamma_\delta} \frac{1}{\xi - w} d\xi \right] + \pi.$$

Next, for every $y \in \partial B(w,\delta) \cap \Omega$, let $\gamma_y \subset \Omega$ be a curve connecting z_0 and $y \in \partial\Omega_\delta \cap \partial B(w,\delta)$, and let $\sigma \subset \partial B(w,\delta) \cap \Omega_\delta$ be chosen so that the domain bounded by $\gamma_\delta + \sigma - \gamma_y$, which is contained in Ω_δ, does not contain w. Then, since the mapping $\xi \mapsto \frac{1}{\xi - w}$ is holomorphic in the domain bounded by $\gamma_\delta + \sigma - \gamma_y$

$$\begin{aligned} 0 &= \int_{\gamma_\delta + \sigma - \gamma_y} \frac{1}{\xi - w} d\xi = \int_{\gamma_\delta} \frac{1}{\xi - w} d\xi + \int_\sigma \frac{1}{\xi - w} d\xi - \int_{\gamma_y} \frac{1}{\xi - w} d\xi \\ &\Rightarrow \mathbb{I}m \left[\int_{\gamma_\delta} \frac{1}{\xi - w} d\xi \right] - 2\pi \leq \mathbb{I}m \left[\int_{\gamma_y} \frac{1}{\xi - w} d\xi \right] \leq \mathbb{I}m \left[\int_{\gamma_\delta} \frac{1}{\xi - w} d\xi \right] + 2\pi, \end{aligned}$$

as the rotation along σ is bounded by the rotation of a circle, which is 2π. Overall, we conclude that

$$\begin{aligned} \left| \mathbb{I}m \left[\int_{\gamma_y} \frac{1}{\xi - w} d\xi \right] - arg_{[w]}(y_\delta - w) \right| &\leq \left| \mathbb{I}m \left[\int_{\gamma_y} \frac{1}{\xi - w} d\xi \right] - \mathbb{I}m \left[\int_{\gamma_\delta} \frac{1}{\xi - w} d\xi \right] \right| \\ &+ \left| \mathbb{I}m \left[\int_{\gamma_\delta} \frac{1}{\xi - w} d\xi \right] - arg_{[w]}(y_\delta - w) \right| \leq 3\pi. \end{aligned}$$

In particular the argument above holds for $y^* \in \partial B(w,\delta) \cap \Omega$ which satisfies $rot(w,\delta) = \exp\left(arg_{[w]}(y^* - w)\right)$. □

The next thing we would like to know is some kind of continuity of the rotation when moving the disc $B(w,\delta)$. As we saw in the previous lemma, estimating the rotation is related to estimating integrals over curves. We will first need a decomposition description of curves:

Proposition 3.5 *Let Γ be a closed curve so that there exist Γ_1, Γ_2 non-self intersecting curves so that $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2$ is precisely the set of points where Γ intersects itself. We orient Γ_j so that Γ is a closed curve. Then there exists closed simple curves $\{\gamma_k\}$ satisfying that*

1. $\bigcup_{k=1}^{N} \gamma_k = \Gamma$.

2. $\bigcup\limits_{i \neq j} [\gamma_i \cap \gamma_j] = \Gamma_1 \cap \Gamma_2$, *which is the set of points where Γ intersects itself.*

3. *For every k for every $j \in \{1, 2\}$ we have $\gamma_k \Big|_{\Gamma_j}$ has the same orientation as Γ_j.*

Proof. Given an intersection point of Γ_1 with Γ_2 there are two pieces of $\Gamma = \Gamma_1 \cup \Gamma_2$ directed towards the intersection point, and two directed outwards. Because Γ_1, Γ_2 are not self intersecting, then one of the pieces directed towards the point belongs to Γ_1 and the other belongs to Γ_2 and the same holds for the pieces directed outwards. We say two pieces are 'companion pieces' if one is directed towards the intersection point and the other is directed outwards and also one belongs to Γ_1 and the other to Γ_2 (see Figure 1).

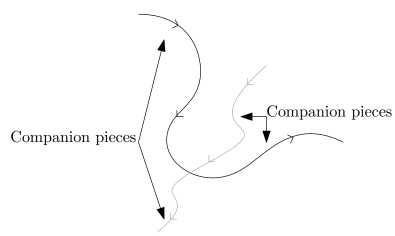
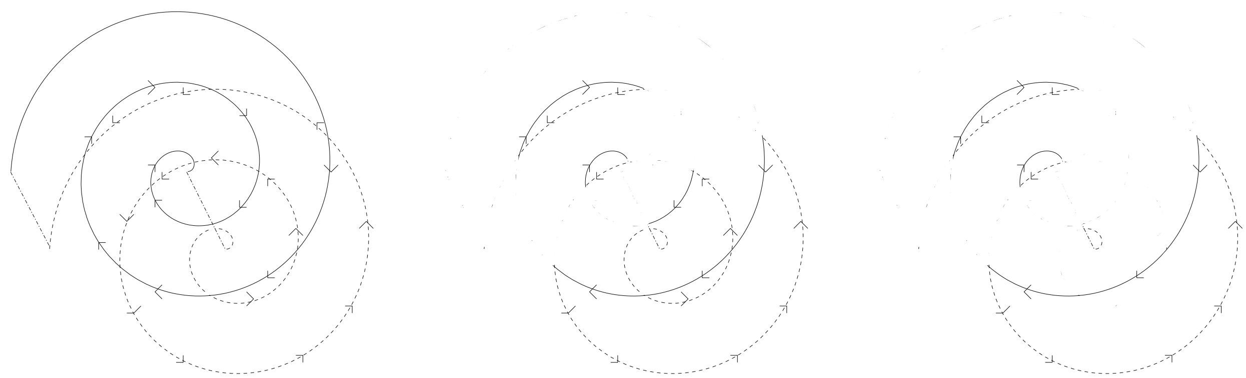


Figure 1: Companion pieces. The grey curve is Γ_1, and the black one is Γ_2.

We will describe an algorithm to create the curves $\gamma_1, \cdots \gamma_N$. We begin at any intersection point of Γ_1 and Γ_2. We choose a curve leaving the intersection point, and follow it according to the orientation assigned to it. At every intersection point, we enter via one curve and we leave the intersection point on its companion curve. Since the curve Γ is closed, at some point we will hit the curve we are creating. The first time encounter an intersection point already in our curve, we will remove from the curve everything that preceded the first visit to that point creating a simple loop, denoted γ_1. Because we removed a closed loop, and the initial curve was closed, we are left with a collection of closed loops, but this time with less intersection points; Every intersection point we used to construct the first curve, where we entered through one curve and left the intersection point via a companion curve, gives rise to an oriented closed curve composed of parts of Γ_1 and Γ_2 like in the original assumption of the proposition. We end up with a collection of closed curves, each composed of a union of two simple curves. We may now apply the algorithm to each one of them to generate $\gamma_2, \cdots, \gamma_N$.

$\square$



(a) Example of a curve. **(b)** After one step. **(c)** After final step.

Figure 2: This figure describes a decomposition of curve composed of a simple curve and its translation.

The following lemma shows that if the center of the target ball, $B(w, \delta)$ is perturbed a little bit, then the rotation does not change by much:

Lemma 3.6 *For every $\xi, w \in \partial\Omega$ if $|\xi - w| < \delta$, then f*

$$|\log(rot(\xi, \delta)) - \log(rot(w, \delta))| \leq 10\pi.$$

Proof. Let Ω_1 denote the connected component of $\Omega \setminus (B(\xi\delta), B(w, \delta))$. Let $\gamma_w, \gamma_\xi \subset \Omega_1$ be two curves connecting z_0 with $\partial B(w, \delta)$ and $\partial B(\xi, z)$ respectively. Let $\sigma \subset \partial B(w, \delta) \cup \partial B(\xi, \delta)$ be so that the domain bounded by $\Gamma := \gamma_w + \sigma - \gamma_w$ does not contain either points. Since $B(w, \delta) \cap B(\xi, \delta) \neq \emptyset$, every curve either circles both points or circles none of them. Now in the domain bounded by Γ both functions $z \mapsto \frac{1}{z-\xi}$ and $z \mapsto \frac{1}{z-w}$ are holomorphic and therefore by the Decomposition Proposition, Proposition 3.5,

$$\int_\Gamma \frac{1}{z - \xi} dz = 0 = \int_\Gamma \frac{1}{z - w} dz,$$

and in particular

$$\left| \mathbb{I}m \left[\int_{\gamma_\xi} \frac{1}{z - w} dz \right] - \mathbb{I}m \left[\int_{\gamma_w} \frac{1}{z - w} dz \right] \right| \leq \left| \mathbb{I}m \left[\int_\sigma \frac{1}{z - w} dz \right] \right| \leq 4\pi.$$

The last piece of the puzzle we need is to observe that since $\gamma_w, \gamma_\xi \subset \Omega_1$ then the number of times each of these curves circles ξ has to be equal to the number of times it circles w for otherwise, the curve separates between the two points, which is impossible by the way Ω_1 was defined. Using the interpretation of the imaginary part of the integral we see that

$$\mathbb{I}m \left[\int_{\gamma_\xi} \frac{1}{z - w} dz \right] = \mathbb{I}m \left[\int_{\gamma_\xi} \frac{1}{z - \xi} dz \right].$$

Overall, using Lemma 3.4 and the estimates above, we see that

$$|\log(rot(B(\xi,\delta)))-\log(rot(B(w,\delta)))|\le\left|\mathbb{I}m\left[\int_{\gamma_\xi}\frac{1}{z-\xi}dz\right]-\log(rot(\xi,\delta))\right|+\left|\mathbb{I}m\left[\int_{\gamma_w}\frac{1}{z-w}dz\right]-\log(rot(w,\delta))\right|$$
$$+\left|\mathbb{I}m\left[\int_{\gamma_\xi}\frac{1}{z-\xi}dz\right]-\mathbb{I}m\left[\int_{\gamma_w}\frac{1}{z-w}dz\right]\right|$$
$$\le 6\pi+\left|\mathbb{I}m\left[\int_{\gamma_\xi}\frac{1}{z-\xi}dz\right]-\mathbb{I}m\left[\int_{\gamma_\xi}\frac{1}{z-w}dz\right]\right|+\left|\mathbb{I}m\left[\int_{\gamma_\xi}\frac{1}{z-w}dz\right]-\mathbb{I}m\left[\int_{\gamma_w}\frac{1}{z-w}dz\right]\right|$$
$$\le 6\pi+0+\left|\mathbb{I}m\left[\int_{\sigma}\frac{1}{z-w}dz\right]\right|\le 10\pi.$$

□

The last lemma we present in this auxiliary subsection is kind of a mean-value theorem for holomorphic functions. While such a theorem is not correct in the original form, a modification of it does hold:

Claim 3.7 *Let I, ζ_I, z_I, ϕ be as in Observation 3.3, i.e., $I\subset\partial\mathbb{D}$ is an arc with $\lambda_1(I)<\frac{1}{2}$, ζ_I is the centre of the arc I, $z_I=\zeta_I(1-\lambda_1(I))$, and $\phi:\mathbb{D}\to\Omega$ is a conformal map. Then there exists a constant $K=K_\Omega$, which depends on the domain Ω alone, and there exists $\eta\in I$ so that*

$$\left|arg_{[\phi(\eta)]}\left[\frac{\phi(z_I)-\phi(\eta)}{z_I-\eta}\right]-arg_{[\phi(\eta)]}\left[\phi'(z_I)\right]\right|\le K_\Omega.$$

The proof of this proposition heavily relies on ideas from the proof of McMillan's twist theorem (see, for example, p.142 in [48]).

Proof. Following Lemma 6.19 in [48], with $z=z_I,\ I$, there exists a point $\eta\in I$ so that

$$|\phi(z_I)-\phi(\eta)|\le K_1\cdot(dist(\phi(z_I),\partial\Omega)+diam(\phi(I))\le K_2\cdot diam(\phi(I)),$$

following Observation 3.3. Let A denote the non-euclidean segment connecting z_I and η, then

$$\int_A|\phi'(\xi)|d|\xi|=|\phi(z_I)-\phi(\eta)|\le K_2\cdot diam(\phi(I)).$$

Let $\alpha\in A$ be so that $\rho_h(\alpha,z_I)=1$, where ρ denotes the hyperbolic distance, and define the non-euclidean segment $\tilde{A}:=\{w\in A,\rho(w,z_I)\ge 1\}$. Then following [48, Cor. 1.5], and Koebe's distortion theorem, for every $z\in\tilde{A}$

$$|\phi(z)-\phi(z_I)|\ge|\phi'(z_I)|\left(1-|z_I|^2\right)\frac{\tanh(\rho(z,z_I))}{4}\ge dist(\phi(z_I),\partial\Omega)\cdot\frac{\tanh(1)}{4}=\frac{dist(\phi(z_I),\partial\Omega)}{K_3},$$

and therefore

$$\int_{\tilde{A}}d\left|arg_{[\phi(\eta)]}\left[\phi(z)-\phi(z_I)\right]\right|\le\int_{\tilde{A}}\frac{|\phi'(z)|}{|\phi(z)-\phi(z_I)|}d|z|\le\frac{K_3}{dist(\phi(z_I),\partial\Omega)}\int_{\tilde{A}}|\phi'(z)|d|z| \tag{3}$$
$$\le\frac{K_3}{dist(\phi(zI),\partial\Omega)}\int_A|\phi'(z)|d|z|\le\frac{K_3}{dist(\phi(z_I),\partial\Omega)}\cdot K_2\cdot dist(\phi(z_I),\partial\Omega)=K_2\cdot K_3.$$

Now define the map $\psi : \mathbb{D} \times \mathbb{D} \to \mathbb{R}$ by

$$\psi(z,w) := \begin{cases} arg_{[\phi(\eta)]}\left(\frac{\phi(z)-\phi(w)}{z-w}\right) & , z \neq w \\ arg_{[\phi(\eta)]}\left(\phi'(z)\right) & , z = w \end{cases}.$$

This map is continuous on $\mathbb{D} \times \mathbb{D}$. In addition, by the triangle inequality

$$|\psi(z_I, z_I) - \psi(\eta, z_I)| \leq |\psi(z_I, z_I) - \psi(\alpha, z_I)| + |\psi(\alpha, z_I) - \psi(\eta, z_I)| = S_1 + S_2.$$

To bound S_1, we use Exercise 1.3(4) in [48]

$$S_1 = |\psi(z_I, z_I) - \psi(\alpha, z_I)| = \left| arg_{[\phi(\eta)]}\left(\phi'(z_I)\right) - arg_{[\phi(\eta)]}\left(\frac{\phi(\alpha) - \phi(z_I)}{\alpha - z_I}\right)\right| \leq 8\rho(z_I, \alpha) + \frac{\pi}{2} \leq 10.$$

To bound S_2 we will use (3),

$$S_2 = |\psi(\alpha, z_I) - \psi(\eta, z_I)| = \int_{\tilde{A}} d\left|arg_{[\phi(\eta)]}\left[\phi(z) - \phi(z_I)\right]\right| \leq K_2 \cdot K_3.$$

Over all, we get that

$$\left| arg_{[\phi(\eta)]}\left[\frac{\phi(z_I) - \phi(\eta)}{z_I - \eta}\right] - arg_{[\phi(\eta)]}\left[\phi'(z_I)\right]\right| = |\psi(\eta, z_I) - \psi(z_I, z_I)| \leq S_1 + S_2 \leq K_\Omega,$$

for some uniform constant K_Ω which depends on the domain alone. ☐

3.1.3 The main Lemma

Lemma 3.8 *Let $\Omega \subset \mathbb{C}$ be a quasidisk. For every disk $B = B(\zeta, \delta)$ centred at $\zeta \in \partial\Omega$, there exists $z \in (1 - \omega(B))\,\mathbb{T}$ satisfying*

(HM) $$\frac{\delta}{\omega(B) \cdot \log^2\left(\frac{1}{\omega(B)}\right)} \lesssim |\phi'(z)| \lesssim \frac{\delta}{\omega(B)}$$ (R) $$\left|\phi'^{-i}(z)\right| \sim rot(B).$$

where the constants depend on the doubling constant of the measure and the domain.

Proof. The proof relies on Koebe's distortion theorem combined with the second part of Observation 3.3 for the harmonic measure and Lemme 3.7 for the rotation.

Let B be a disk. Note that by Carleson's lemma, there exists a continuum of diameter δ, $\beta \in \partial\Omega \cap 2B$, satisfying that $\omega(\beta) \geq \frac{\omega(B)}{\log^2\left(\frac{1}{\omega(B)}\right)}$. However, since the harmonic measure is doubling, by looking at the continuation of β in $3B$ we may assume that $diam(\beta) \in (\delta, 6\delta)$ and

$$\frac{\omega(B)}{\log^2\left(\frac{1}{\omega(B)}\right)} \leq \omega(\beta) \leq C\omega(B).$$

Let $\phi : \mathbb{D} \to \Omega$ be a Riemann map, and let $z := z_\beta\,(1 - \omega(B))$ where z_β is the centre of the arc $\phi^{-1}(\beta)$. Note that

$$\rho(z, z_\beta\,(1 - \omega(\beta))) \sim \frac{|\omega(B) - \omega(\beta)|}{\min\left(\omega(B), \omega(\beta)\right)} \begin{cases} \gtrsim 1 \\ \lesssim \log^2\left(\frac{1}{\delta}\right) \end{cases}.$$

Since $\log\phi'$ and a Bloch function, we see that

$$\log|\phi'(z)| \sim \log|\phi'(z_\beta(1-\omega(\beta))|(1+o(1)) \sim \frac{d_\phi(z_\beta(1-\omega(\beta)))}{1-|z_\beta(1-\omega(\beta))|}(1+o(1)) \sim \frac{diam(\phi(\beta))}{\omega(\beta)}(1+o(1)).$$

following the second part of Observation 3.3, since Ω assumed to be a quasi-disk.

To show the second half of the lemma, we will first show that

$$\frac{rot(B)}{|\phi'^{-i}(z)|} \sim_\Omega 1.$$

Let $\eta \in \phi^{-1}(\beta)$ be the point from Claim 3.7, satisfying that $|\phi(\eta)-\zeta| < diam(\phi(\beta))$ while

$$\left|arg_{[\phi(\eta)]}\left[\frac{\phi(z)-\phi(\eta)}{z-\eta}\right] - arg_{[\phi(\eta)]}[\phi'(z)]\right| \le K_\Omega.$$

We need to relate the argument $arg_{[\phi(\eta)]}[\phi(z)-\phi(\eta)]$ with $rot(\phi(\eta),\delta)$ as it is not necessarily the case that $\phi(z) \in \partial B(\phi(\eta),\delta)$.

Following Observation 3.3,

$$|\phi(z)-\phi(\eta)| \le dist(\phi(z),\beta) + diam(\beta) \le M\cdot\delta,$$

for some constant $M = M(\Omega)$, which depends on the domain alone. Let $M_k := 2M\cdot\frac{diam(\beta)}{\delta} \sim_\Omega 1$. Then there exists a sequence of tangential disks $\{B_\ell\}_{\ell=1}^{M_k}$ so that $B_\ell = B(\xi_\ell,\delta)$ with $\xi_1 = \phi(\eta)$ while $\phi(z) \in \partial B_{M_k}$. Let $\sigma_\ell \subset \partial B_\ell \cup \partial B_{\ell+1}$ be so that $\sum_{\ell=1}^{M_k}\sigma_\ell$ is a curve in Ω connecting $\partial B_1 \cap \Omega$ with $\phi(z)$, let $\gamma_{\phi(\eta)}, \gamma_{\phi(z)} \subset \Omega$ be two curves connecting z_0 with ∂B_1 and ∂B_{M_k} respectively. Note that the domain bounded by the curves $\gamma_{\phi(z)} - \gamma_{\phi(\eta)} + \sum_{\ell=1}^{M_k}\sigma_\ell$ does not contain the point $\phi(\eta)$. Then, a similar argument to the one presented in Lemma 3.6 shows that

$$\begin{aligned}&\left|\log(rot(\phi(\eta),\delta)) - arg_{[\phi(\eta)]}[\phi(z)-\phi(\eta)]\right| \le \left|\mathbb{I}m\left[\int_{\gamma_{\phi(\eta)}}\frac{1}{\xi-\phi(\eta)}d\xi\right] - \mathbb{I}m\left[\int_{\gamma_{\phi(\eta)}+\sum_{\ell=1}^{M_k}\sigma_\ell}\frac{1}{\xi-\phi(\eta)}d\xi\right]\right| + 5\pi \\ &\le 5\pi + \sum_{\ell=1}^{M_k}\left|\mathbb{I}m\left[\int_{\sigma_\ell}\frac{1}{\xi-\phi(\eta)}d\xi\right]\right| \le \pi(M_k+5),\end{aligned}$$

since the change in the argument along each σ_ℓ is bounded by 2π. Next,

$$\begin{aligned}\left|\log(rot(B)) - arg_{[\phi(\eta)]}[\phi'(z)]\right| &\le |\log(rot(B)) - \log(rot(\phi(\eta),\delta))| \\ &+ \left|\log(rot(\phi(\eta),\delta)) - arg_{[\phi(\eta)]}[\phi(z)-\phi(\eta)]\right| \\ &+ \left|arg_{[\phi(\eta)]}[\phi(z)-\phi(\eta)] - arg_{[\phi(\eta)]}[\phi'(z)]\right| \lesssim_\Omega 1,\end{aligned}$$

as the first summand is bounded following Lemma 3.6, the second summand is bounded by the computation above, and the third summand is bounded by Claim 3.7. Over all, we get that

$$\left|\phi'^{-i}(z)\right| \sim rot(B),$$

concluding the proof. □

Remark 3.9 *Note that if $3B' \cap 3B = \emptyset$, are two disks of the same harmonic measure, then by definition,*

$$\rho(z,z') \sim \frac{|z-z'|}{\omega(B)} \sim 1.$$

In particular, the points are distinct.

3.2 The proof of Theorem 2.1

Proof. It is enough to show that for every $\varepsilon > 0$ there exists a sequence $\{r_k\} \nearrow 1$ so that for every k large enough,

$$\frac{\log\left(\lambda_1\left(L_{a-\varepsilon,b-\varepsilon}(r_k)\right)\right)}{\log\left(\frac{1}{1-r_k}\right)} \geq (1-a)\, f_\Omega\left(\frac{1}{1-a}, \frac{-b}{1-a}\right) - 1 - \varepsilon.$$

Fix $\varepsilon > 0$ and let $\eta \in \left(0, \frac{\varepsilon\cdot\alpha}{3}\right)$ and $\{\delta_k\}$ be so that $\lim\limits_{k\to\infty} \frac{\log(N(\delta_k,\alpha,\gamma,\eta))}{\log\left(\frac{1}{\delta_k}\right)} \geq f_\Omega(\alpha,\gamma) - \varepsilon\cdot\alpha$. For every k there exists a collection of disjoint disks $\left\{B_j^k\right\}_{j=1}^{N(\delta_k,\alpha,\gamma,\eta)}$ of radius δ_k satisfying properties 1-4 in the definition of $N(\delta_k,\alpha,\gamma,\eta)$. By excluding at most a linear portion of the disks in the collection, we may assume without loss of generality that $3B_j^k \cap 3B_\nu^k = \emptyset$ for every $j \neq \nu$. Following Lemma 3.8, if the harmonic measure of Ω is doubling, then for every j there exists $z_j \in (1-\delta_k^\alpha)\,\mathbb{T}$ so that (HM) and (R) hold. In fact, those two hold for $z_j' = \left(1-\omega(B_j^k)\right)\zeta_j$, however,

$$\rho(z_j,z_j') \sim \frac{\left|z_j - z_j'\right|}{\min\left\{\omega(B_j^k),\delta_k^\alpha\right\}} \leq \delta_k^\eta \cdot \frac{\left|\omega(B_j^k)-\delta_k^\alpha\right|}{\delta_k^{\alpha-2\eta}} = \delta_k^{3\eta}\left|1-\frac{\omega(B_j^k)}{\delta^{\alpha-2\eta}}\right| \leq \delta_k^{3\eta},$$

therefore

$$\left|\log\phi'(z_j) - \log\phi'(z_j')\right| \lesssim C$$

for some uniform constant C.

Note that in this case, if δ_k is small enough (depending on η)

$$\delta_k^{1-\alpha+2\eta} \leq \frac{\delta}{C_\Omega \delta_k^\alpha \log^2\left(\frac{1}{\delta_k^\alpha}\right)} \leq |\phi'(z_j)| \leq C_\Omega \frac{\delta_k}{\delta_k^\alpha} \leq \delta^{1-\alpha-2\eta}.$$

Similarly

$$\delta_k^{\gamma+2\eta} \leq \frac{rot(B)}{C_\Omega} \leq \left|\phi'^{-i}(z)\right| \leq C_\Omega rot(B) \leq \delta_k^{\gamma-2\eta}.$$

We divide $(1-\delta_k^\alpha)\,\mathbb{T}$ into $N := \lceil \delta_k^{-\alpha} \rceil$ arcs of equal length, and denote this collection $\mathcal{P}_k$. Note that for every $j \neq \nu$ we have $\rho(z_j,z_\nu) \sim 1$ so by excluding at most a linear portion of the disks, the points $\{z_j\}$ belong to different arcs in this collection.

Given r_0 we set $r_k = 1-\delta_k^\alpha$, for δ_k small enough so that $r_k > r_0$, and note that for every arc $I \in \mathcal{P}_k$ if $z_j \in I$ for some j, then for every $z \in I$

$$|\phi'(z)| \in \left(\delta_k^{1-\alpha+3\eta}, \delta_k^{1-\alpha-3\eta}\right) \quad , \quad e^{Arg(\phi'(z))} = \left|\phi'^{-i}(z)\right| \in \left(\delta_k^{\gamma+3\eta}, \delta_k^{\gamma-3\eta}\right).$$

Next,

$$\delta_k^{1-\alpha\pm3\eta} = (\delta_k^\alpha)^{1-\frac{1}{\alpha}\pm\frac{3\eta}{\alpha}} = (1-r_k)^{1-\frac{1}{\alpha}\pm\frac{3\eta}{\alpha}} \text{ and } \delta_k^{\gamma\pm3\eta} = (\delta_k^\alpha)^{\frac{\gamma}{\alpha}\pm\frac{3\eta}{\alpha}} = (1-r_k)^{\frac{\gamma}{\alpha}\pm\frac{3\eta}{\alpha}}$$

implying that

$$\frac{\log|\phi'(z)|}{\log\left(\frac{1}{1-r_k}\right)} \in \left(1-\frac{1}{\alpha}-\frac{3\eta}{\alpha}, 1-\frac{1}{\alpha}+\frac{3\eta}{\alpha}\right) = (a-\varepsilon, a+\varepsilon)$$
$$\frac{Arg\left(\phi'(z)\right)}{\log\left(\frac{1}{1-r_k}\right)} \in \left(\frac{\gamma}{\alpha}-\frac{3\eta}{\alpha}, \frac{\gamma}{\alpha}+\frac{3\eta}{\alpha}\right) = (b-\varepsilon, b+\varepsilon).$$

Then

$$\lambda_1\left(L_{a-\varepsilon,b-\varepsilon}(r_k)\right) \geq (1-r_k)\,\#\left\{z_j\right\} \gtrsim (1-r_k)\,N(\delta_k,\alpha,\gamma,\eta) \gtrsim (1-r_k)\cdot\delta_k^{-(f_\Omega(\alpha,\gamma)-\varepsilon\cdot\alpha)}$$
$$= (1-r_k)^{1-\frac{f_\Omega(\alpha,\gamma)}{\alpha}+\varepsilon},$$

implying that

$$\frac{\log\left(\lambda_1\left(L_{a-\varepsilon,b-\varepsilon}(r_k)\right)\right)}{\log\left(\frac{1}{1-r_k}\right)} + 1 \geq \frac{f_\Omega(\alpha,\gamma)}{\alpha} - \varepsilon$$

concluding the proof. □

4 Counter Examples

In this section we will prove two counter examples. The first shows that it is not always the case that the Minkowski distortion spectrum dwarfs the Minkowski dimension spectrum. The second one shows that the Minkowski distortion spectrum does not even necessarily dominate the dimension of the set of of points with the correct corresponding lower density. We begin by proving auxiliary results that will be used in both examples.

4.1 Auxiliary Results for the examples

4.1.1 A General Construction

All the examples we present begin with a smooth shape like a disk or a smoothed out square (will be defined below), to generate a sequence of smooth domains that converge to the domain we are after. In both cases we use various 'tubes'. We begin by describing this 'smoothing' mechanism and the tubes.

Let R be a rectangle of length ℓ_k and width $\omega_k < \ell_k$. We define the 'smoothing' of R as the result of the following process- remove the 4 triangles at the corners of R and replace each triangle by a quarter of a disk of radius $\frac{\omega_k}{2}$ (see Figure 3). We will denote the resulting shape by $S(R)$ and refer to it as 'the tube of R'. We will apply a similar process when attaching two tubes to one another or to a domain.

When we connect two 'tubes' a 'smoothed' cube forms their connection. This 'smoothed' cube has edge length ω_k and if one rescales it the resulting shape is exactly the same, giving us uniform bounds on the harmonic measure of parts of this cube (see Figure 4).

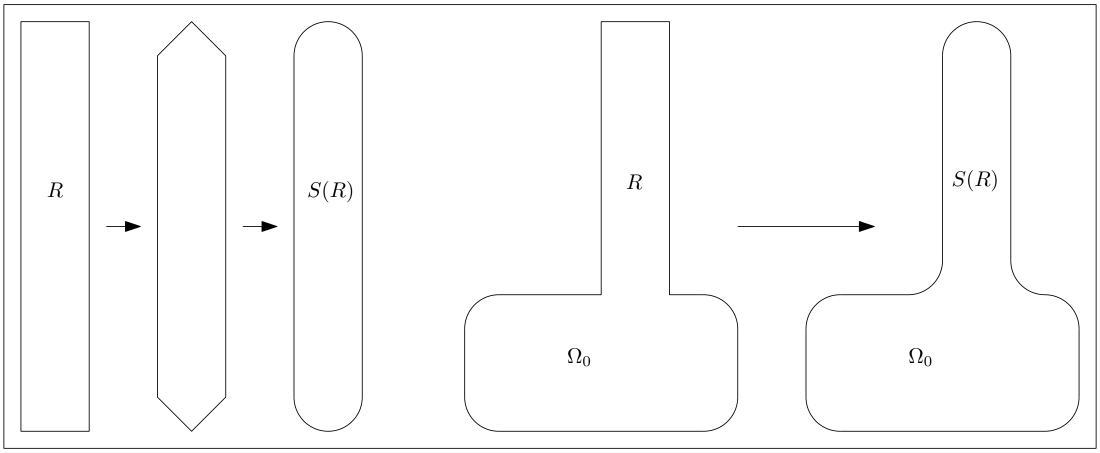


Figure 3: Smoothing of rectangles: The left figure shows the smoothing of one rectangle. The right figure shows the smoothing of a rectangle connected to a domain.

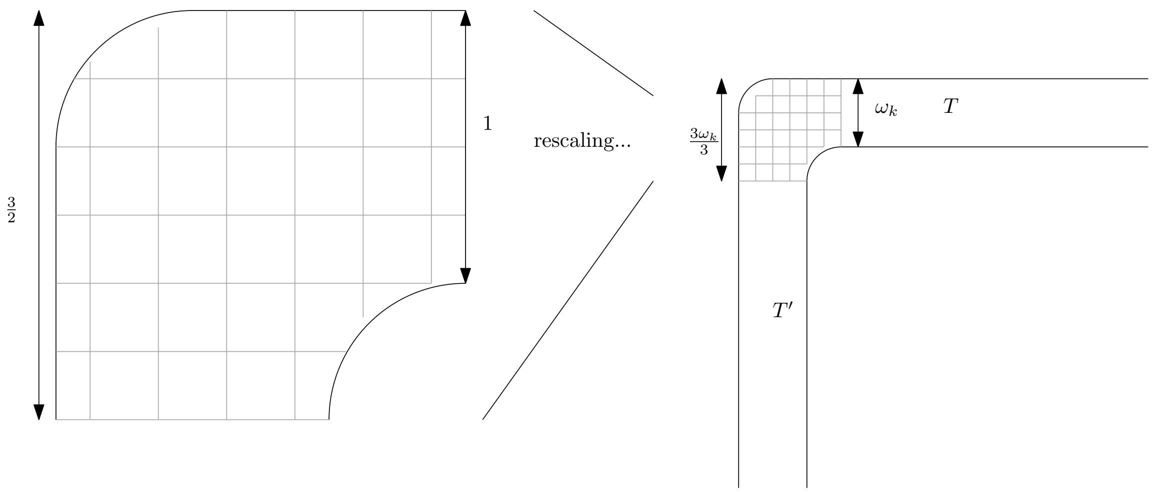


Figure 4: Connecting two tuned we get a smoothed out cube. This cube is a rescaling of the same object.

4.1.2 Estimates of harmonic measures

We shall prove a general estimate on the harmonic measure of curves inside a chain of tubes. Naturally this estimate will be a relative estimate conditioned on the probability to get to the 'entrance' of the tube.

Lemma 4.1 *Let Ω be a simply connected domain that contains a sequence of tubes $T_1, T_2, \cdots, T_m$ of scale k, connected to one another, i.e., T_j and T_{j+1} share a smoothed cube. We allow T_m to be shorter but require it to be longer than ω_k. We denote by $J \subset \partial T_1$ the 'entrance' to the sequence of tubes, of width ω_k, and define Ω_0 as the connected component of $\Omega \setminus J$ that contains z_0 (see Figure 5).*

Given $\gamma \subset \bigcup_{j=1}^{N} \partial T_j$ we denote by $\ell(\gamma)$ the length of γ and define 'the height of γ' in the chain of tubes, $h(\gamma)$, by

$$h(\gamma) = \begin{cases} dist(\gamma, J) & , \gamma \cap \partial T_1 \neq \emptyset \\ (j-1)\cdot \ell_k + dist(\gamma, \partial T_{j-1}) & , \forall \nu \leq j-1,\ \gamma \cap \partial T_\nu = \emptyset \text{ and } \gamma \cap \partial T_j \neq \emptyset \end{cases}.$$

Let γ be a curve so that there exists an arc $A \subset \mathbb{T}$ with $\phi(A) = \gamma$ and $\lambda_1(A) \ll \ell(\gamma)$. Then

$$\omega(z_0, \gamma; \Omega) \lesssim \begin{cases} e^{\pi\cdot m} \exp\left(-\pi \cdot \frac{h(\gamma)}{\omega_k}\right) \cdot \omega(z_0, J; \Omega_0) & , \ell(\gamma) \geq \frac{\omega_k}{100} \\ \frac{\ell(\gamma)}{\omega_k} \cdot e^{\pi\cdot m} \exp\left(-\pi \cdot \frac{h(\gamma)}{\omega_k}\right) \cdot \omega(z_0, J; \Omega_0) & , otherwise \end{cases},$$

and

$$\omega(z_0, \gamma; \Omega) \gtrsim \begin{cases} e^{-\pi\cdot m} \exp\left(-\pi \cdot \frac{h(\gamma)}{\omega_k}\right) \cdot \omega\left(z_0, \frac{1}{2}J; \Omega_0\right) & , \ell(\gamma) \geq \frac{\omega_k}{100} \\ \frac{\ell(\gamma)}{\omega_k} \cdot e^{-\pi\cdot m} \exp\left(-\pi \cdot \frac{h(\gamma)}{\omega_k}\right) \cdot \omega\left(z_0, \frac{1}{2}J; \Omega_0\right) & , otherwise \end{cases},$$

where the constants are all numerical constants.

Proof. Assume that $h(\gamma) > \omega_k$ and denote by J_γ the interval beginning at height $h(\gamma) - \omega_k$ orthogonal to $\partial\Omega$, and let ζ_γ be the midpoint of this interval (see Figure 5 below). We define the auxiliary domain Ω_γ as the connected component of $\Omega \setminus J_\gamma$ containing z_0. Note that $\Omega_0 \subset \Omega_\gamma \subset \Omega$, and that $\Omega_\gamma \setminus \Omega_0$ is a union of tubes (see Figure 5).

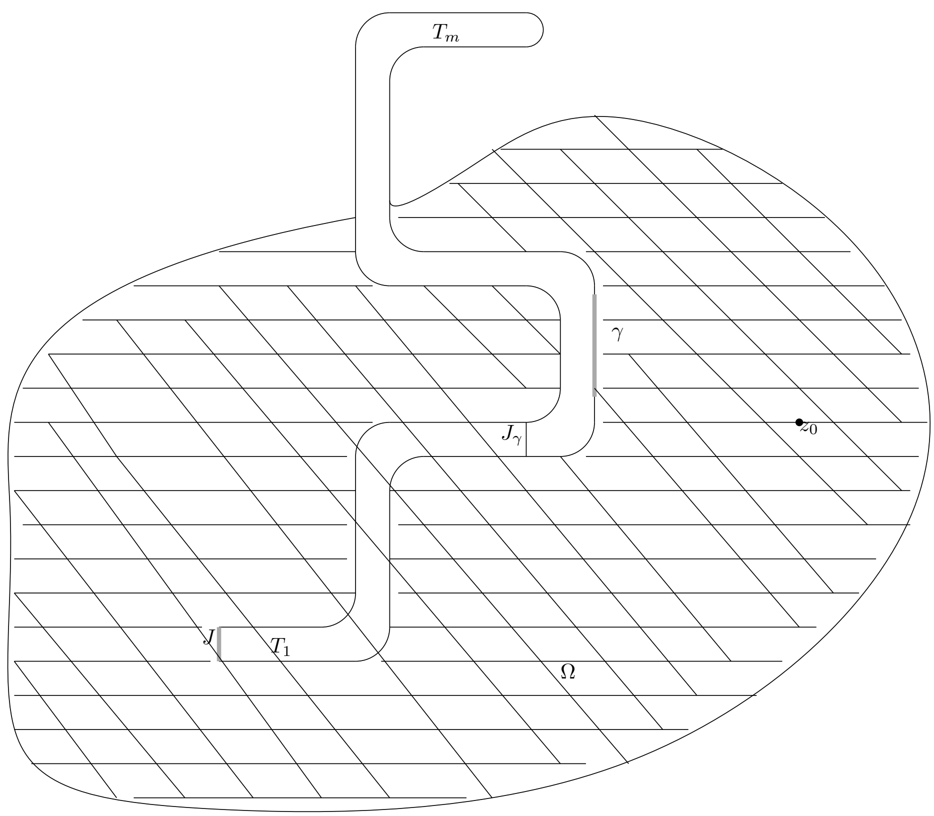


Figure 5: The horizontal lines depict the domain Ω_0, the slanted lines depict the domain Ω_γ. Lastly, γ and J are marked as thick gray lines.

The map $z \mapsto \omega(z,\gamma;\Omega)$ is harmonic in Ω and therefore in Ω_γ

$$\omega(z_0,\gamma;\Omega) = \int\limits_{\partial\Omega_\gamma} \omega(\zeta,\gamma;\Omega) d\omega(z_0,z;\Omega_\gamma) = \int\limits_{J_\gamma} \omega(\zeta,\gamma;\Omega) d\omega(z_0,z;\Omega_\gamma).$$

Note that if $\ell(\gamma) \gtrsim \omega_k$ then for every $\zeta \in J_\gamma$ we have $\omega(\zeta,\gamma;\Omega) \sim 1$ by Beurling. Otherwise, we will consider the upper and lower bounds for $\omega(\zeta,\gamma;\Omega)$, $\zeta \in J_\gamma$ separately.

Upper Bound: For every $\zeta \in J_\gamma$, we will get an upper bound by using extremal length. As we need an upper bound on the harmonic measure, we need to bound from below $\lambda(\zeta,\gamma)$. Let σ be the line connecting ζ and $\partial\Omega \setminus \gamma$ orthogonal to $\partial\Omega$ but in the opposite direction to where γ lies. Let x_γ be the beginning of the curve γ and define the metric

$$\rho(z) := \frac{1}{|z - x_\gamma|} \cdot \mathbf{1}_{B(x_\gamma,\omega_k)\setminus B(x_\gamma,\ell(\gamma))}(z).$$

It is non-negative and well defined. Next, the curve connecting x_γ to the point ζ by a straight line is in the collection $\Gamma(\sigma,\gamma)$ and the function $z \mapsto \frac{1}{|z-x_\gamma|} \cdot \mathbf{1}_{B(x_\gamma,\omega_k)\setminus B(x_\gamma,\ell(\gamma))}(z)$ attains all the values between $\frac{1}{\omega_k}$ and $\frac{1}{\ell(\gamma)}$ once (since the distance between ζ and the curve is at least ω_k), and therefore

$$L^2(\Gamma(\sigma,\gamma),\rho) \le \left(\int_{\ell(\gamma)}^{\omega_k} \frac{1}{t} dt\right)^2 = \log^2\left(\frac{\omega_k}{\ell(\gamma)}\right).$$

On the other hand, note that for disks centered at x_γ, $B(x_\gamma, R)$ for every $0 < r < R$ we have $\lambda_1\left(\Omega \cap B(x_\gamma,r)\right) \le \pi \cdot r$ (even smaller if we include one of the semi-cubes) and therefore

$$A(\Omega,\rho) = \int_{\Omega\cap(B(x_\gamma,w_k)\setminus B(x_\gamma,\ell(\gamma)))} \rho(z) dm(z) \le \pi \int_{\ell(\gamma)}^{\omega_k} r \cdot \frac{1}{r^2} dr = \pi \log\left(\frac{\omega_k}{\ell(\gamma)}\right),$$

implying that

$$\lambda(\zeta,\gamma) \ge \frac{L^2(\Gamma(\sigma,\gamma),\rho)}{A(\Omega,\rho)} \ge \frac{\log^2\left(\frac{\omega_k}{\ell(\gamma)}\right)}{\pi \log\left(\frac{\omega_k}{\ell(\gamma)}\right)}$$

and in turn for every $\zeta \in J_\gamma$,

$$\omega(\zeta,\gamma;\Omega) \le \frac{8}{\pi} \exp\left(-\pi\lambda(\zeta,\gamma)\right) \le \frac{8}{\pi} \cdot \exp\left(-\pi \cdot \frac{\log^2\left(\frac{\omega_k}{\ell(\gamma)}\right)}{\pi \cdot \log\left(\frac{\omega_k}{\ell(\gamma)}\right)}\right) \sim \frac{\ell(\gamma)}{\omega_k}.$$

Overall,

$$\omega(z_0,\gamma;\Omega) \lesssim \frac{\ell(\gamma)}{\omega_k} \cdot \omega(z_0, J_\gamma;\Omega_\gamma).$$

Lower Bound: Note that by inclusion and Harnack's inequality,

$$\begin{aligned}\omega(z_0,\gamma;\Omega) &= \int\limits_{J_\gamma} \omega(\zeta,\gamma;\Omega) d\omega(z_0,z;\Omega_\gamma) \ge \int\limits_{\frac{1}{2}J_\gamma} \omega(\zeta,\gamma;\Omega) d\omega(z_0,z;\Omega_\gamma) \\ &\sim \omega(\zeta_\gamma,\gamma;\Omega) \cdot \omega\left(z_0, \frac{1}{2}J_\gamma;\Omega_\gamma\right) \ge \omega(\zeta_\gamma,\gamma;R_\gamma) \cdot \omega\left(z_0, \frac{1}{2}J_\gamma;\Omega_\gamma\right) \sim \frac{\ell(\gamma)}{\omega_k} \cdot \omega\left(z_0, \frac{1}{2}J_\gamma;\Omega_\gamma\right)\end{aligned}$$

where $R_\gamma \subset \Omega$ is a rectangle of width ω_k and length in $(\omega_k, 2\omega_k)$ (depending on the location of γ with respect to the connected smoothed cubes).

We conclude that

$$\begin{cases} \omega(z_0,\gamma;\Omega) \sim \omega(z_0, J_\gamma;\Omega_\gamma) & ,\ell(\gamma) \geq \frac{\omega_k}{100} \\ \omega(z_0,\gamma;\Omega) \begin{cases} \lesssim \frac{\ell(\gamma)}{\omega_k} \cdot \omega(z_0, J_\gamma;\Omega_\gamma) \\ \gtrsim \frac{\ell(\gamma)}{\omega_k} \cdot \omega\left(z_0, \frac{1}{2}J_\gamma;\Omega_\gamma\right) \end{cases} & ,\text{otherwise} \end{cases}.$$

It is left to bound $\omega(z_0, J_\gamma;\Omega_\gamma)$ from above and $\omega\left(z_0, \frac{1}{2}J_\gamma;\Omega_\gamma\right)$ bellow. As before, the map $z \mapsto \omega(z,\gamma;\Omega_\gamma)$ is harmonic in Ω_γ and therefore in Ω_0. Then

$$\omega(z_0, J_\gamma;\Omega_\gamma) = \int\limits_{\partial\Omega_0} \omega(\zeta, J_\gamma;\Omega_\gamma) d\omega(z_0,\zeta;\Omega_0) = \int\limits_J \omega(\zeta, J_\gamma;\Omega_\gamma) d\omega(\zeta_0, z;\Omega_0).$$

Upper Bound: For every $\zeta \in J$, we will get an upper bound by using extremal length. As we need an upper bound on the harmonic measure, we need to bound from below $\lambda(\zeta, J_\gamma)$. Let σ be the line connecting ζ and $\partial\Omega_\gamma \setminus J_\gamma$ along J. Using the serial rule, if $\Gamma_j := \{\mu \cap T_j; \mu \in \Gamma\}$ where T_j is the j'th tunnel, then

$$\lambda_{\Omega_\gamma\setminus\sigma}(\Gamma) \geq \sum_{j=1}^m \lambda_{T_j}(\Gamma_j) \geq \frac{h(\gamma) - m\cdot\omega_k}{\omega_k} = \frac{h(\gamma)}{\omega_k} - m.$$

as the tunnels T_j become disjoint once we remove the smoothing cubes connecting them, while the extremal length of a rectangle is known. This implies that for every $\zeta \in J$

$$\omega(\zeta, J_\gamma;\Omega_\gamma) \leq e^{\pi\cdot m} \cdot \exp\left(-\pi \cdot \frac{h(\gamma)}{\omega_k}\right).$$

Overall, we see that

$$\omega(z_0, J_\gamma;\Omega_\gamma) = \int\limits_J \omega(\zeta, J_\gamma;\Omega_\gamma) d\omega(\zeta_0, z;\Omega_0) \leq \frac{8}{\pi} \cdot e^{\pi\cdot m} \exp\left(-\pi \cdot \frac{h(\gamma)}{\omega_k}\right) \cdot \omega(z_0, J;\Omega_0).$$

Lower Bound: Let ζ_J denote the center of the interval J and let C_γ be the connected component of $\Omega_\gamma \setminus \left(J \cup \zeta_J + \left[-\frac{\omega_k}{2}, \frac{\omega_k}{2}\right]^2\right)$ which contains J_γ in its boundary. By inclusion and, using Harnack's inequality,

$$\begin{aligned} \omega\left(z_0, \frac{1}{2}J_\gamma;\Omega_\gamma\right) &= \int\limits_{\partial\Omega_0} \omega\left(\zeta, \frac{1}{2}J_\gamma;\Omega_\gamma\right) d\omega(z_0,\zeta;\Omega_0) = \int\limits_J \omega\left(\zeta, \frac{1}{2}J_\gamma;\Omega_\gamma\right) d\omega(z_0,\zeta;\Omega_0) \\ &\geq \frac{1}{4} \cdot \omega\left(z_0, \frac{1}{2}J;\Omega_0\right) \cdot \omega\left(\zeta_J, \frac{1}{2}J_\gamma;\Omega_\gamma\right) \geq \frac{1}{4} \cdot \omega\left(z_0, \frac{1}{2}J;\Omega_0\right) \cdot \omega\left(\zeta_J, \frac{1}{2}J_\gamma;C_\gamma\right). \end{aligned}$$

To bound the later, we will use extremal length, copying the proof done for rectangles. Note that if $\sigma \cap (C_\gamma \setminus \Omega_0) \neq \emptyset$ then $\lambda_{C_\gamma\setminus\sigma}(\Gamma(\sigma,\gamma))$ becomes smaller. As we are looking for an upper bound, and we are taking supremum over all such curves, we may consider only curves which do not intersect $(C_\gamma \setminus \Omega_0)$.

Let ρ be any metric on C_γ. Fix a point $w \in \partial C_\gamma \setminus \partial\Omega_\gamma$, and denote by γ_w the curve running parallel to the boundary of Ω_γ starting from w and ending on J_γ. Then for every such w there exists $\mu \in \Gamma(\sigma,\gamma)$ so that $\mu \subset \gamma_w$.

Then,

$$L^2(\Gamma(\sigma,\gamma),\rho) = \left(\inf_{\mu\in\Gamma(\sigma,\gamma)} \int\limits_{\mu} \rho(\zeta) d\,|\zeta|\right)^2 \leq \left(\int_{\gamma_w} \rho(\zeta) d|\zeta|\right)^2 \leq (h(\gamma) + m\cdot\omega_k)\cdot \int_{\gamma_w} \rho^2(\zeta) d|\zeta|$$

by Cauchy-Schwarts inequality. Integrating along J we get

$$\begin{aligned}\omega_k \cdot L^2(\Gamma(\sigma,\gamma),\rho) &= \omega_k \cdot \left(\inf_{\mu\in\Gamma(\sigma,\gamma)} \int\limits_{\mu} \rho(\zeta) d\,|\zeta|\right)^2 \leq \int_0^{\omega_k} \left(\int_{\gamma_w} \rho(\zeta) d|\zeta|\right)^2 dw \\ &\leq (h(\gamma) + m\cdot\omega_k) \int_{\Omega_\gamma \setminus \Omega_0} \rho^2(\zeta) dm(\zeta) = (h(\gamma) + m\cdot\omega_k)\, A(\Omega_\gamma \setminus \Omega_k^0, \rho) \leq (h(\gamma) + m\cdot\omega_k)\, A(C_\gamma, \rho).\end{aligned}$$

This implies that for every σ connecting ζ_k with $\partial C_\gamma \setminus J_\gamma$ outside of $(C_\gamma \setminus \Omega_0)$,

$$\lambda_{C_\gamma\setminus\sigma}(\Gamma(\sigma,\gamma)) \leq \frac{h(\gamma) + m\cdot\omega_k}{\omega_k}.$$

Lastly, using symmetry we see that

$$\omega\left(\zeta_J, \frac{1}{2}J_\gamma; \Omega_\gamma\right) \geq \omega\left(\zeta_J, \frac{1}{2}J_\gamma; C_\gamma\right) \geq \exp\left(-\pi\lambda(\zeta_J,\gamma)\right) \geq e^{-\pi\cdot m}\cdot \exp\left(-\pi\frac{h(\gamma)}{\omega_k}\right).$$

Combining everything together the proof follows.

The case where $h(\gamma) < \omega_k$ should be discussed. However, copying the proof estimating $\omega(\zeta,\gamma;\Omega)$ for $\zeta \in J$ concludes the proof of this case as well. □

The second lemma in this subsubsection gives a lowers bound for the minimal length of some curves. The idea is that since these components are smooth, then if ε is too small then $\omega(\gamma) \sim \ell(\gamma) \gg \varepsilon^\beta$.

Lemma 4.2 *Let ε be so that there exists $\gamma \in \bigcup_{j=1}^m \partial T_j$ so that $\omega(\gamma) = \varepsilon^\beta$ and $\ell(\gamma) \geq \varepsilon$. If $\ell(\gamma) < \min\left\{\frac{\omega_k}{100}, \omega(z_0, J; \Omega_0)^{\frac{1}{\beta}}\right\}$, then*

$$\varepsilon \gtrsim \left(\frac{\omega\left(z_0, \frac{1}{2}J; \Omega_0\right)}{\omega_k} \cdot \exp\left(-\pi\cdot m\left(\frac{\ell_k}{\omega_k} + 1\right)\right)\right)^{\frac{1}{\beta-1}}.$$

Proof. Recall that the longest height inside components in $\bigcup_{j=1}^m \partial T_j$ is bounded by $m\cdot\ell_k$. Following the estimate done in Lemma 4.1, we see that for some uniform constant $C > 1$

$$\begin{aligned}\varepsilon^\beta = \omega(\gamma) &\geq \frac{\ell(\gamma)}{C} \cdot \frac{\omega\left(z_0, \frac{1}{2}J; \Omega_0\right)}{\omega_k} \cdot e^{-\pi\cdot m} \exp\left(-\pi\cdot\frac{h(\gamma)}{\omega_k}\right) \geq \frac{\varepsilon}{C} \cdot \frac{\omega\left(z_0, \frac{1}{2}J; \Omega_0\right)}{\omega_k} \cdot e^{-\pi\cdot m} \exp\left(-\pi\cdot\frac{h(\gamma)}{\omega_k}\right) \\ &\geq \frac{\varepsilon}{C} \cdot \frac{\omega\left(z_0, \frac{1}{2}J; \Omega_0\right)}{\omega_k} \cdot \exp\left(-\pi\cdot m\left(\frac{\ell_k}{\omega_k} + 1\right)\right),\end{aligned}$$

implying that

$$\varepsilon \geq \left(\frac{\omega\left(z_0, \frac{1}{2}J; \Omega_0\right)}{C\cdot\omega_k} \cdot \exp\left(-\pi\cdot m\left(\frac{\ell_k}{\omega_k} + 1\right)\right)\right)^{\frac{1}{\beta-1}}$$

concluding the proof. □

4.2 Example 1: Dimension Spectrum vs. Distortion Spectrum

In this section we will prove Theorem 2.2. We will construct for every $a \in (0,1)$ a domain, Ω, satisfying that $(1-a)\, f_\Omega\left(\frac{1}{1-a}\right) \geq \frac{1-a}{2}$ while $d_\Omega(a) < \frac{1-a}{2}$ showing that Theorem 2.2 does not hold in general and that the additional requirement the Ω is a quasi-disk is necessary.

4.2.1 The Construction:

Let $z_0 := -\frac{1}{2}$, let $\{n_k\} \subset \mathbb{N}$ be a subsequence of the natural numbers that will be chosen later. For every k we let $\delta_k = 2^{-2n_k}, \ell_k := \sqrt{\delta_k},\ \omega_k = \frac{\sqrt{\delta_k}}{\nu \cdot n_k} = \frac{2^{-n_k}}{\nu \cdot n_k}$ for some $\nu > 1$ that will be chosen later as well. We define the sequence of intervals:

$$I_k := \left\{(x,y),\ x \in \left[\sqrt{\delta_k} - \omega_k, \sqrt{\delta_k}\right]\ , y = -\sqrt{\delta_k}\right\}$$
$$U_k := \left\{(x,y),\ x \in \left[\sqrt{\delta_k} - \omega_k, \sqrt{\delta_k}\right],\ y = \delta_k \sqrt{1 - \frac{\delta_k^{2\left(\frac{1}{1-a}-1\right)}}{4}}\right\}.$$

Let ℓ_b be a smooth line connecting the origin with $\partial\mathbb{D}$ satisfying that $\bigcup_{k=1}^{\infty} I_k \subset \ell_b$ and let ℓ_u be a smooth line connecting the origin with $\partial\mathbb{D}$ satisfying that $\bigcup_{k=1}^{\infty} U_k \subset \ell_u$ and $\ell_b \cap \ell_u = \emptyset$. We denote by Ω_0 the set whose boundary is composed of ℓ_b, ℓ_u and $\partial\mathbb{D}$ which contains z_0 (see Figure 6), and we choose n_1 small enough so that for every j,

$$\frac{\ell(I_j)}{2} \leq \omega(z_0, I_j; \Omega_0) \leq 2\ell(I_j).$$

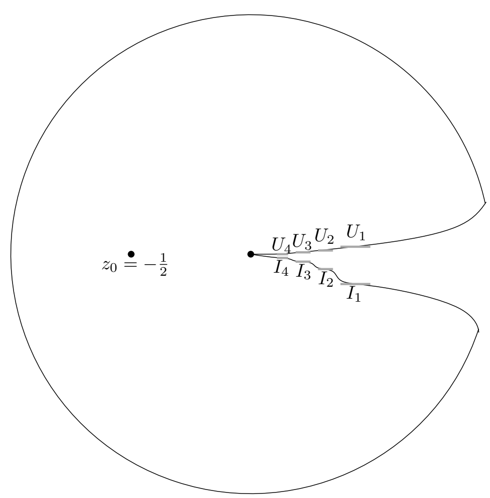


Figure 6: The initial set Ω_0. The grey lines at the top are U_k, the grey lines at the bottom are I_k.

For every k we denote by T_k the smoothing of $I_k \times \left[-\sqrt{\delta_k}, 0\right]$, i.e., $T_k = S\left(I_k \times \left[-\sqrt{\delta_k}, 0\right]\right)$, and let $\Omega = \Omega_0 \cup \bigcup_{k=1}^{\infty} T_k$. For every interval I we will denote by $t \cdot I$ the interval of length $t \cdot \ell(I)$ concentric with I, and for every k we let $\Omega_k := \Omega \setminus T_k$, and let J_k denote the straight part of the upper edge of T_k, i.e. $J_k = \partial T_k \cap \{\mathbb{I}m\,[z] = 0\}$.

4.2.2 The proof

Following lemma 3.2 part 1, it is enough to bound the number of curves in $\Gamma(a', r)$ for all $a' > a$ and r with $(1-r)$ small enough. Given r we define $\varepsilon := (1-r)^{1-a'}$. We will bound the number of such curves in each scale, k. Fix k and let us look at three cases:

Case 1- $\varepsilon > 2^{-n_k}$: For every k,

$$\ell(T_k) = \omega_k + 2\sqrt{\delta_k} \le 3 \cdot 2^{-n_k},$$

therefore for every k fixed

$$\ell\left(\partial\Omega \cap 2^{-n_k}\mathbb{D}\right) \le \ell\left(\left[0, 2^{-n_{k+1}}\right]\right) + \sum_{j=k+1}^{\infty} \ell(T_j) \le 2^{-n_{k+1}} + \sum_{j=k+1}^{\infty} 3 \cdot 2^{-n_j} \le 7 \cdot 2^{-n_{k+1}} < 2^{-n_k},$$

if n_k is chosen so that $7 \cdot 2^{-n_k} < 2^{-n_{k-1}}$. We get that the number of curves of diameter at least ε in $\partial\Omega \cap 2^{-n_k}\mathbb{D}$ is at most 2, i.e. if $\varepsilon > 2^{-n_k}$ then the set $\partial\Omega \cap 2^{-n_k}\mathbb{D}$ is covered by at most two disjoint curves of length at least ε.

Case 2- $\frac{\omega_k}{n_k} \le \varepsilon < 2^{-n_k}$: Then the number of disjoint curves in $\Gamma(a', r)$ in the tube T_k is bounded by

$$\frac{\ell(\partial T_k)}{\varepsilon} \le \frac{2\left(\ell_k + \omega_k\right)}{\frac{\omega_k}{n_k}} \lesssim \nu \cdot n_k^2 \lesssim \nu \cdot \log^2\left(\frac{1}{\varepsilon}\right).$$

Case 3- $\varepsilon < \frac{w_k}{n_k}$: Following Lemma 4.2,

$$\varepsilon \gtrsim \left(\frac{\omega\left(z_0, \frac{1}{2}I_k; \Omega_0\right)}{\omega_k} \cdot \exp\left(-\pi \cdot m\left(\frac{\ell_k}{\omega_k} + 1\right)\right)\right)^{\frac{1}{a'-1}} \sim \exp\left(-\frac{\pi \cdot \nu}{a'-1} \cdot \log\left(\frac{1}{\ell_k}\right)\right) = \ell_k^{\frac{\pi \cdot \nu}{a'-1}}$$

since $\{n_k\}$ were chosen so that $\omega\left(z_0, \frac{1}{2}I_k; \Omega_0\right) \sim \ell(I_k) = \omega_k$, and $m = 1$. This implies that the number of disjoint curves in $\Gamma(a', r)$ in the tube T_k is bounded by

$$\frac{\ell(\partial T_k)}{\varepsilon} \lesssim \frac{\ell_k}{\varepsilon} \lesssim \varepsilon^{\frac{a'-1}{\pi \cdot \nu} - 1}.$$

Let $k_1 := \max\left\{k, \varepsilon < 2^{-n_k}\right\}$, $k_2 := \max\left\{k, \varepsilon < \frac{w_k}{n_k}\right\}$, then there exists a constant C so that

$$\begin{aligned}
\#\Gamma(a', r) &\le 2 + \sum_{j=1}^{k_2} \#\left\{\gamma \in \Gamma(a', r), \gamma \subset T_j \text{ disjoint curves}\right\} \\
&\le 2 + \sum_{j=1}^{k_1} \#\left\{\gamma \in \Gamma(a', r), \gamma \subset T_j \text{ disjoint curves}\right\} + \sum_{j=k_1+1}^{k_2} \#\left\{\gamma \in \Gamma(a', r), \gamma \subset T_j \text{ disjoint curves}\right\} \\
&\lesssim k_1 \cdot \nu \cdot \log^2\left(\frac{1}{\varepsilon}\right) + \sum_{j=k_1+1}^{k_2} \frac{\ell_k}{\varepsilon} \cdot \mathbf{1}_{\left\{\ell_j^{\frac{\pi \cdot \nu}{a'-1}} \le C\varepsilon\right\}} \le k_1 \cdot \log^2\left(\frac{1}{\varepsilon}\right) + (k_2 - k_1) \cdot \varepsilon^{\frac{\beta-1}{\pi \cdot \nu} - 1} \le \log^2\left(\frac{1}{\varepsilon}\right) \cdot \varepsilon^{\frac{a'-1}{\pi \cdot \nu} - 1}
\end{aligned}$$

for ε numerically small enough.

Following lemma 3.2 we conclude that

$$\begin{aligned}
d_{\Omega}^{curve}(a) &= \limsup_{a' \searrow a} \limsup_{r \nearrow 1} \frac{\log\left(\#\Gamma(a', r)\right)}{\log\left(\frac{1}{1-r}\right)} \le \limsup_{a' \searrow a} \limsup_{r \nearrow 1} \frac{\log\left(\#\Gamma(a', r)\right)}{a' \cdot \log\left(\frac{1}{\varepsilon}\right)} \\
&\le \limsup_{a' \searrow a} \limsup_{r \nearrow 1} \frac{\log\left(\log^2\left(\frac{1}{\varepsilon}\right) \cdot \varepsilon^{\frac{a'-1}{\pi \cdot \nu} - 1}\right)}{a' \cdot \log\left(\frac{1}{\varepsilon}\right)} = \frac{1 + \frac{1}{\pi \cdot \nu} - \frac{a'}{\pi \cdot \nu}}{a'} = (1 - a')\left(1 + \frac{1}{\pi \cdot \nu}\right) - \frac{1}{\pi \cdot \nu} < \frac{1-a}{2},
\end{aligned}$$

if ν is small enough (depending on a).

It is left to bound $f_\Omega\left(\frac{1}{1-a}\right)$ from bellow. Fix k and let $\left\{z_j^k\right\}_{j=1}^{M_k}$ be the maximal collection of points on J_k satisfying that for every $i \neq j,\ |z_i - z_j| > 2\delta_k$. Then

1. The discs $B\left(z_j^k, \delta_k\right)$ are disjoint.

2. By the way ℓ_u was defined, for every j, $\omega\left(B\left(z_j^k, \delta_k\right)\right) \sim length(U_k \cap B(z_j^k, \delta_k) \sim \delta_k^{\frac{1}{1-a}}$.

We conclude that for every η there exists k large enough so that $N\left(\delta_k, \frac{1}{1-a}, \eta\right) \geq M_k$. On the other hand

$$M_k \geq \frac{\frac{4}{5}\cdot\omega_k}{2\delta_k} = \frac{2\cdot\frac{\sqrt{\delta_k}}{\nu\cdot n_k}}{5\delta_k} \gtrsim \frac{1}{\sqrt{\delta_k}\log\left(\frac{1}{\delta_k}\right)}.$$

This implies that

$$f_\Omega\left(\frac{1}{1-a}\right) = \lim_{\eta\to 0}\limsup_{\delta\to 0}\frac{\log\left(N\left(\delta, \frac{1}{1-a}, \eta\right)\right)}{\log\left(\frac{1}{\delta}\right)} \geq \lim_{k\to\infty}\frac{\log(M_k)}{\log\left(\frac{1}{\delta_k}\right)} \geq \frac{1}{2},$$

concluding the proof of Theorem 2.2.

4.3 Example 2: Minkowski distortion spectrum vs. Hausdorff dimension

4.3.1 The Construction:

4.3.1.1 The set of density: In this subsection, we will show that for every $\alpha \in \left(1, \frac{3}{2}\right)$ and for every $c \in (1, \alpha)$ there exists a set C_α of dimension $\frac{1}{\alpha}$, a domain Ω, a sequence of scales, $\{\delta_k\} \searrow 0$, and a sequence of errors, $\{\eta_k\} \searrow 0$ so that for every $z \in C_\alpha$ and every k we have

$$\delta_k^{\frac{\alpha}{c}+\eta_k} \leq \omega(z_0, B(z, \delta_k); \Omega) \leq \delta_k^{\frac{\alpha}{c}-\eta_k}$$

however,

$$d_\Omega\left(1 - \frac{c}{\alpha}\right) \leq d_\Omega^{curve}\left(1 - \frac{c}{\alpha}\right) < \frac{c}{\alpha^2} \leq \frac{c}{\alpha} d_{\mathcal{H}}\left(C_\alpha\right).$$

Throughout this section we will use the notation $a_k \sim b_k$ if

$$\lim_{k\to\infty}\frac{\log(a_k)}{\log(b_k)} = 1.$$

For every sequence $\{n_k\} \subset \mathbb{N}$ we define the α-Cantor set in the following inductive way:

Step 0: Let $C_0 = [0,1]$, $\mathcal{C}_0 := \{[0,1]\}$.

Step 1: Split the interval C_0 into 2^{n_1} subintervals of equal length, take every forth interval I and denote by I_α the interval beginning at the same point at I of length $\ell(I_\alpha) := \ell(I)^\alpha$. We denote the collection of intervals be $\mathcal{C}_1$ and define the set $C_1 = \bigcup_{I_\alpha \in C_1} I_\alpha$. The set C_1 is composed of 2^{n_1-2} intervals of length $2^{-\alpha\cdot n_1}$ each.

Step k: Split every interval $I \in \mathcal{C}_{k-1}$ into 2^{n_k} subintervals, take every forth interval and define the collection $\mathcal{C}_k$ to

be all the intervals I_α that originated from $\mathcal{C}_{k-1}$, and the set $C_k := \bigcup_{I_\alpha \in \mathcal{C}_k} I_\alpha$. The set C_k is composed of $2^{\sum_{j=1}^{k}(n_j-2)}$ intervals of length $2^{-\alpha \sum_{j=1}^{k} n_j}$ composing the collection $\mathcal{C}_k$. For brevity define $N_k := \sum_{j=1}^{k} n_j$.
We then define by $C_\alpha := \bigcap_{k=1}^{\infty} C_k$. It is a well define subset of the interval $[0,1]$.

Observation 4.3 *The set C_α defined above has dimension $\frac{1}{\alpha}$.*

4.3.1.2 The domain Ω: As in the previous example, we will construct the set Ω as a monotone, this time decreasing, limit of sets Ω_k. We begin with the set $\Omega_0 := S\left([-1,1]^2\right) \setminus [0,1]$.
For every k we let $\ell_k := 2^{-c \cdot N_k}$ and $\omega_k := \frac{\ell_k}{\nu \cdot \log\left(\frac{1}{\ell_k}\right)}$ where $\nu > 0$ will be chosen at the very end of the proof to be a small constant.

Step 1: For every $I \in \mathcal{C}_1$ we place a cubic annulus of outer edge-length ℓ_1 and inner edge-length $\ell_1 - w_1$ such that I is at the bottom of the outer cube, distance $\frac{\ell_1}{4}$ from the right hand border of the annulus and trim the annulus at the end of the interval I, and leave a gate of width ω_1 at the entrance (see Figure 7).

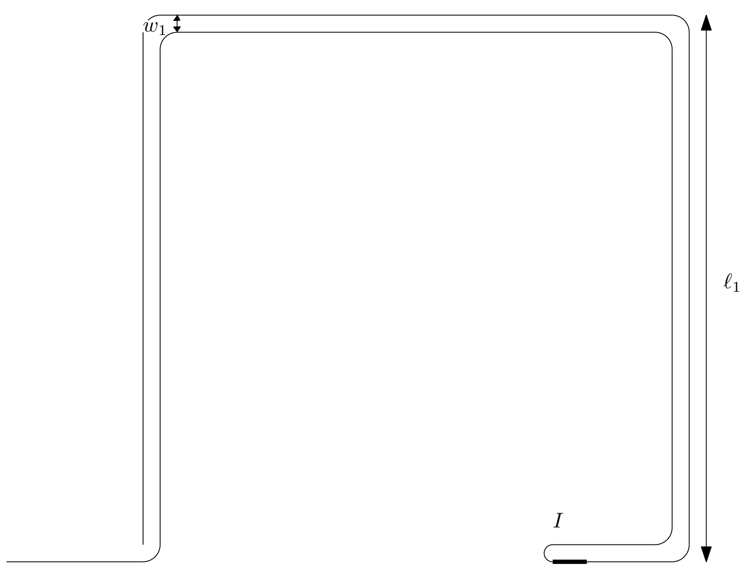


Figure 7: Step 1 of the construction.

Step k: For every $I \in \mathcal{C}_k$ we place a cubic annulus of outer edge-length ℓ_k and inner edge-length $\ell_k - \omega_k$ such that I is at the bottom of the outer cube, distance $\frac{\ell_k}{4}$ from the right hand border of the annulus and trim the annulus at the end of the interval I and leaving only a small gate of width ω_k at the entrance (see Figure 8). Denote the connected components of $\partial\Omega_k \setminus \partial\Omega_{k-1}$ by C_k^j, we have 2^{n_k} such components inside each copy C_{k-1}^j. We choose n_k large enough so that the entire annulus fits inside the tube about the parent interval of I in step $k-1$.

We define Ω_k be the connected component of $\Omega_{k-1} \setminus \bigcup_{j=1}^{2^{N_k}} \partial C_k^j$, which contains z_0. Then $\Omega_k \subseteq \Omega_{k-1}$, and therefore $\Omega := \bigcap_{k=1}^{\infty} \Omega_k$ is well defined and non empty, as it includes z_0.

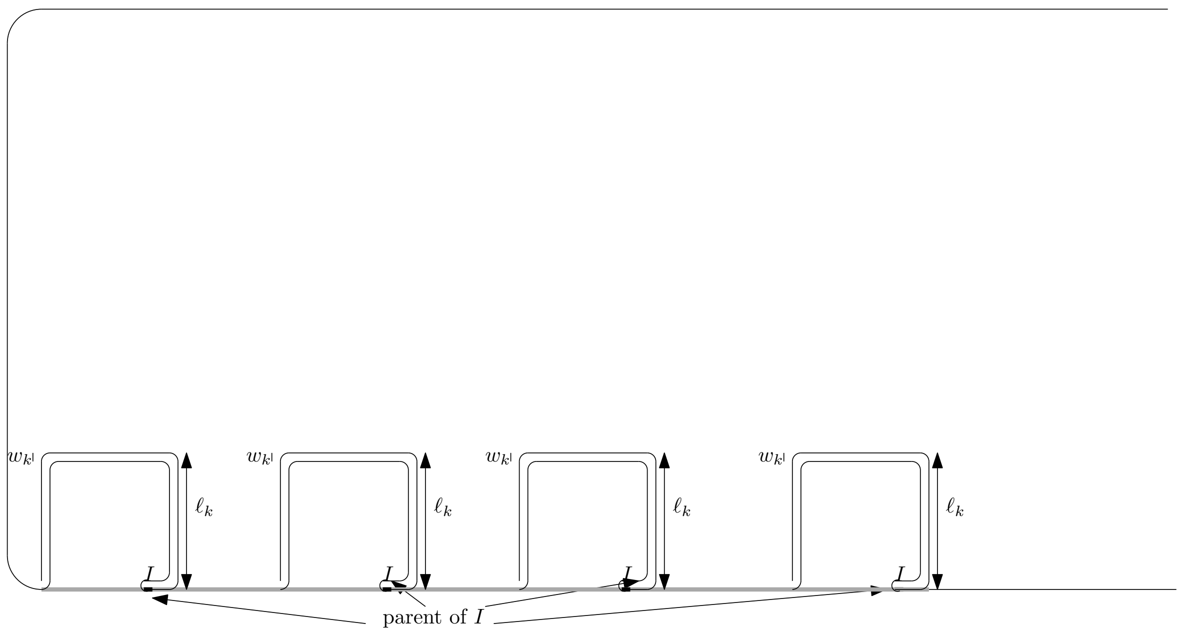


Figure 8: Step k of the construction. The thick gray line is the parent interval of all the I's from step $(k-1)$.

On every step k we smooth-out the boundary like in the previous counter example so that if n_k is chosen large enough, then for every curve $\gamma \subset \partial\Omega_{k-1}$ with $\ell(\gamma) < \ell_k$ we have

$$\ell(\gamma)^{1+\eta_k} \le \omega(\gamma) \le \ell(\gamma)^{1-\eta_k},$$

where η_k are chosen so that $\frac{\alpha}{c} > 1 + \eta_k$ and $\{\eta_k\} \searrow 0$.

Lastly, as Ω is a simply connected set, we shall denote by $\phi : \mathbb{D} \to \Omega$ the conformal map, which maps $\mathbb{D}$ onto Ω. Recall that λ_1 almost surely, ϕ can be extended to $\partial\mathbb{D}$.

Note that if we set $\delta_k := \frac{\ell_k}{4} + 5 \cdot 2^{-\alpha \cdot N_k}$ then for every $z \in C_\alpha$ we have that

$$\omega(z_0, B(z, \delta_k); \Omega) \sim \omega(z_0, B(z, \delta_k) \setminus \partial C_k; \Omega) \sim \ell(B(z, \delta_k) \setminus \partial C_k) \sim 2^{-\alpha \cdot N_k},$$

implying that

$$\delta_k^{\frac{\alpha}{c}+\eta_k} \le \omega(z_0, B(z, \delta_k); \Omega) \le \delta_k^{\frac{\alpha}{c}-\eta_k}$$

as stated above.

4.3.2 The proof

In light of Lemma 3.2 part 1, the goal now is to bound the number of curves in the collection $\Gamma(a', r)$ for every $a' > a = 1 - \frac{c}{\alpha}$ and every r with $(1-r)$ small enough.

Fix $a' > a$ and r, and let $\varepsilon := (1-r)^{1-a'}$. Define k_ε be so that $\ell_{k_\varepsilon+1} \le \varepsilon < \ell_{k_\varepsilon}$. Then, for every $j \le k_\varepsilon - 1$ every curve $\gamma \subset \partial\Omega_j$ of length at least ε contains a curve γ' of length exactly $\varepsilon < \ell_{k_\varepsilon}$ which satisfies that

$$\omega(\gamma) \ge \omega(\gamma') \ge \ell(\gamma')^{1+\eta_k} = \varepsilon^{1+\eta_k} \gg (1-r),$$

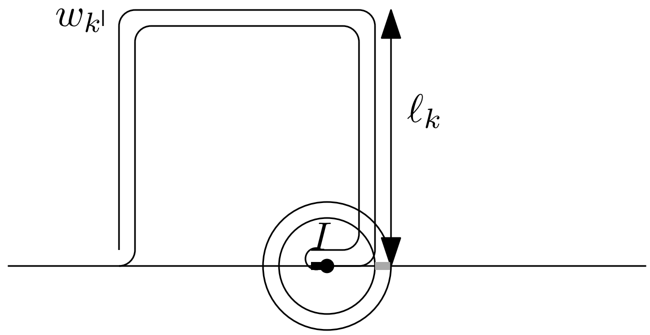


Figure 9: The smaller disk has radius $\geq \frac{\ell_k}{4}$ but it picks up a very small harmonic measure. The gray line while having small length, $\sim 2^{-\alpha \cdot N_k}$, is what dominates the harmonic measure.

by the way we chose n_k, and η_k. This implies that for such ε for every $\gamma \in \Gamma(a', r)$, the intersection of γ with $\partial\Omega_j$ must have length at most $(1-r) \ll (1-r)^{1-a'}$ as $a' > 0$, and therefore at most two curves will be contained in $\partial\Omega_j$ and we will count it there.

On the other hand, for every $j \geq k_\varepsilon + 1$ for every curve $\gamma \subset \partial\Omega_j \setminus \partial\Omega_{k_\varepsilon}$ of length at least ε, by using Lemma 4.2, with $m = 4$,

$$\omega_{k+1}^{1-\eta_k} \geq \varepsilon^{\frac{1}{1-a'}} = \omega(\gamma) \gtrsim \left(\omega_k^{\eta_k} \exp\left(-4\pi\left(\frac{\ell_k}{\omega_k} + 1\right)\right)\right)^{\frac{1}{a'}}$$
$$\geq \left(\ell_k^{\eta_k(1+o(1))} \exp\left(-4\pi \cdot \nu \log\left(\frac{1}{\ell_k}\right)(1+o(1))\right)\right)^{\frac{1}{a'}} \geq \ell_k^{\frac{1}{a'}(1+o(1))(\eta_k + 4\pi\cdot\nu)}$$

which is impossible if n_{k+1} is chosen large enough. We see that for such r for every $\gamma \in \Gamma(a', r)$, the intersection of γ with $\partial\Omega_j \setminus \partial\Omega_{k_\varepsilon}$ must intersect $\partial\Omega_{k_\varepsilon}$ and therefore it is enough to count it there.

It is left to bound how many such curves are in $\partial\Omega_{k_\varepsilon} \setminus \partial\Omega_{k_\varepsilon - 1}$. We will first bound the number of 'long' curves, curves with $\ell(\gamma) \geq \frac{\omega_k}{100}$. However, like in the first example, since $\frac{\ell_k}{\omega_k} \sim \log\left(\frac{1}{\ell_k}\right)$ we see that the number of such curves is (up to multiplication by a constant) $\log\left(\frac{1}{\ell_k}\right)$.

To bound the number of 'short' curves, we will use the calculation done for the second case,

$$(1-r) = \omega(\gamma) \geq \cdots \gtrsim \ell_k^{\frac{1}{a'}(1+o(1))(\eta_k + 4\pi\cdot\nu)} \Rightarrow \frac{\log\left(\frac{1}{\ell_k}\right)}{\log\left(\frac{1}{1-r}\right)} \geq \frac{1}{a'(4\pi\cdot\nu + \eta_k)}(1-o(1)).$$

This implies that

$$d_\Omega^{curve}(a) = \limsup_{a' \searrow a} \limsup_{r \nearrow 1} \frac{\log(\#\Gamma(a', r))}{\log\left(\frac{1}{1-r}\right)}$$
$$\leq \limsup_{a' \searrow 1-\frac{c}{\alpha}} \limsup_{r \nearrow 1} \frac{\log\left(2^{N_k}\left(\log\left(\frac{1}{\ell_k}\right) + \frac{\ell_k}{(1-r)^{1-a'}}\right)\right)}{\log\left(\frac{1}{1-r}\right)} \leq \limsup_{a' \searrow 1-\frac{c}{\alpha}} \limsup_{r \nearrow 1} \frac{\log\left(\frac{\ell_k^{1-\frac{1}{c}}}{(1-r)^{1-a'}}\right)}{\log\left(\frac{1}{1-r}\right)}(1+o(1))$$
$$= \frac{1-\frac{1}{c}}{\frac{\alpha}{c}} \limsup_{a' \searrow 1-\frac{c}{\alpha}} \left(1 - \limsup_{r \nearrow 1} \frac{\log\left(\frac{1}{\ell_k}\right)}{(1-a')\log\left(\frac{1}{1-r}\right)}(1-o(1))\right) \leq \frac{c-1}{\alpha}\left(1 - \frac{\frac{\alpha}{c} - 1}{4\pi\cdot\nu}\right) < \frac{c}{\alpha^2}$$

as long as $\nu > 0$ is chosen small enough depending on α and c (in fact, by choosing ν small we can make this as small as we wish). This concludes the proof of Theorem 2.3.

5 Approximations

5.1 Thermodynamical Multifractal formalism

5.2 Approximating with polygons

Observation 5.1 *It is enough to show:*

(1) $F^{+}(\alpha) = \sup_{F \text{ IFS}} f^{+}_{\Omega_F}(\alpha)$ *where* $F^{+}(\alpha) := \sup_{\Omega} f^{+}_{\Omega}(\alpha)$.

(2) $\sup_{F \text{ IFS}} d_{\Omega_F}(a) = \sup_{\Omega} d_{\Omega}(a)$ *for all* $a > 0$.

Proof. Recall that $f(\alpha) = \min\{f^{-}(\alpha), f^{+}(\alpha)\}$ therefore $F(\alpha) \le \min\{F^{+}(\alpha), F^{-}(\alpha)\} \le F^{+}(\alpha)$. If $F^{+}(\alpha) = \sup_{F \text{ IFS}} f^{+}_{\Omega_F}(\alpha)$, then

$$\sup_{F \text{ IFS}} f_{\Omega_F}(\alpha) \le F(\alpha) \le F^{+}(\alpha) = \sup_{F \text{ IFS}} f^{+}_{\Omega_F}(\alpha).$$

However, for finite iterated functions systems, $f^{+}_{\Omega_F}(\alpha) = f^{-}_{\Omega_F}(\alpha)$, implying that

$$F(\alpha) = F^{+}(\alpha) = \sup_{F \text{ IFS}} f^{+}_{\Omega_F}(\alpha).$$

Next, recall that for iterated function systems, Carleson't estimate, Lemma 5.14, shows that the harmonic measure of Ω_F is doubling, hence satisfies the requirements of Theorem 2.1. Then

$$F(\alpha) = F^{+}(\alpha) \overset{\text{by (1)}}{=} \sup_{F \text{ IFS}} f_{\Omega_F}(\alpha) \overset{\substack{\text{for IFS} \\ \text{Thm 2.1}}}{=} \sup_{F \text{ IFS}} \alpha \cdot d_{\Omega_F}\left(1 - \frac{1}{\alpha}\right) \overset{\text{by (2)}}{=} \sup_{\Omega} \alpha \cdot d_{\Omega}\left(1 - \frac{1}{\alpha}\right),$$

concluding the proof. □

Theorem 5.2 *Let $\eta > 0$ and $\Omega_0 \subseteq \mathbb{C}$ be any bounded symmetric simply connected domain. Let $\phi_0 : \mathbb{D} \to \Omega_0$ be a Riemann mapping sending 0 to z_0 with $|\phi_0'(0)| = 1$. Let $n \in \mathbb{N}$ be large enough (depending on Ω_0 and η) and define $\phi(z) := \phi_0\left(\left(1 - \frac{1}{n}\right) z\right)$ and $\Omega_1 := \phi(\mathbb{D})$. Divide $\mathbb{T}$ into n arcs of equal length, and let $\{\gamma_k\}$ be the image of these arcs under ϕ.*

Given a sub-collection $\left\{\gamma_{\kappa_j}\right\}_{j=1}^{m}$ yours to choose there exists a collection of disks covering the boundary of a horizontally symmetric polygon, P, satisfying that

1. *(a) For every disk in the collection, D, $\#\left\{B, B \cap \frac{3}{2}D \neq \emptyset\right\} = 3$.*

   (b) For every disk in the collection, D, $\partial P \cap D$ is a line segment.

   (c) The disks intersecting the real axis and their neighbours have radius $\frac{1}{n^4}$.

2. *There exists a partition of the collection of disks, $\mathcal{P} = \{P_k\}$, where P_k is associated with the curve γ_k and for at least half of the elements in the collection $\{\gamma_{\kappa_j}\}_{j=1}^{m}$ the associated collection contains*

   (a) *a disk with $\omega(z_0, D; P) \geq \frac{1}{2n^{1+4\eta}}$, and $r(D) < diam(\gamma_k)$.*

   (b) *a disk with $\omega(z_0, D; P) \leq \frac{2}{n^{1-2\eta}}$, and $r(D) > diam(\gamma_k)^{1+\eta}$.*

   where $r(D)$ denotes the radius of the disk D.

The proof of the theorem will have three main parts-

1. Modify the boundary of Ω_1 on some part of the collection $\{\gamma_{\kappa_j}\}_{j=1}^{m}$.
2. Show that this modification, does not change much the harmonic measure of at least half of the elements in the collection $\{\gamma_{\kappa_j}\}$.
3. Cover the boundary of the approximation by disks to create a polygon, and show they satisfy the requirements of the Theorem.

5.2.1 Step 1: The construction:

Let $\{z_k\}$ denote the endpoints of the arcs in the partition of $\mathbb{T}$ into n arcs of equal length. Note that for every k,

$$\omega(z_0, \gamma_k; \Omega_1) = \omega(0, \phi^{-1}(\gamma_k); \mathbb{D}) = \frac{1}{n}.$$

We cover the curve γ_{κ_j} with disjoint sub-curves of harmonic measure between $\frac{1}{n^{1+\eta}}$ and $\frac{2}{n^{1+\eta}}$ (if the last curve has harmonic measure less than $\frac{1}{n^{1+\eta}}$ we re-define the second to last curve to be their union). There are at most

$$\frac{\omega(z_0, \gamma_{\kappa_j}, \Omega_1)}{\frac{1}{n^{1+\eta}}} = \frac{\frac{1}{n}}{\frac{1}{n^{1+\eta}}} = n^{\eta},$$

such curves, and in particular, one of these sub-curves has diameter greater than $diam(\gamma_{\kappa_j}) \cdot n^{-\eta}$, denote its endpoints $z_{\kappa_j} \leq a_j < b_j \leq z_{\kappa_j+1}$. Let $\tilde{\gamma}_{\kappa_j}$ denote the curve obtained by replacing this sub-curve with the line segment $[\phi(a_j), \phi(b_j)]$, and let Ω_1^j be the domain obtained from Ω_1 by replacing γ_{κ_j} with $\tilde{\gamma}_{\kappa_j}$. We then define Ω_2 as the domain obtained by replacing γ_{κ_j} with $\tilde{\gamma}_{\kappa_j}$ for all $1 \leq j \leq m$. The boundary of Ω_2 is not self intersecting, see Claim 5.8 below.

5.2.2 Step 2: The harmonic measure of half of the elements $\{\gamma_{\kappa_j}\}$ does not change much:

Lemma 5.3 *At least half of the curves in the collection $\{\gamma_{\kappa_j}\}_{j=1}^{m}$ satisfy that $\omega(z_0, \tilde{\gamma}_k; \Omega_2) \in \left(\frac{1}{n^{1+2\eta}}, \frac{1}{n^{1-2\eta}}\right)$.*

Proof. For every $1 \le j \le m$ and $\ell \in \mathbb{N}$ we define the collections

$$M_\ell^+(j) := \left\{\nu; 1 - \frac{\omega(z_0, \gamma_{\kappa_j}; \Omega_1^\nu \cap \Omega_1)}{\omega(z_0, \gamma_{\kappa_j}; \Omega_1)} \in \left[2^{-\ell-1}, 2^{-\ell}\right)\right\} \;,\; M_\ell^-(j) := \left\{\nu; \frac{\omega(z_0, \gamma_{\kappa_j}; \Omega_1^\nu \cup \Omega_1)}{\omega(z_0, \gamma_{\kappa_j}; \Omega_1)} - 1 \in \left[2^{-\ell-1}, 2^{-\ell}\right)\right\}$$
$$D_\ell^+(j) := \left\{\nu; 1 - \frac{\omega(z_0, \gamma_{\kappa_\nu}; \Omega_1^j \cap \Omega_1)}{\omega(z_0, \gamma_{\kappa_\nu}; \Omega_1)} \in \left[2^{-\ell-1}, 2^{-\ell}\right)\right\} \;,\; D_\ell^-(j) := \left\{\nu; \frac{\omega(z_0, \gamma_{\kappa_\nu}; \Omega_1^j \cup \Omega_1)}{\omega(z_0, \gamma_{\kappa_\nu}; \Omega_1)} - 1 \in \left[2^{-\ell-1}, 2^{-\ell}\right)\right\}.$$

In a sense, $M_\ell^\pm(j)$ is the collections of all curves that disturbed the harmonic measure of the j'th sub-curve by a factor of $2^{-\ell}$, and the collection $D_\ell^\pm(j)$ is the collection of continua that the modification to the j'th sub-curve disturbs by a factor of $2^{-\ell}$. Note that, by definition,

$$\nu \in M_\ell(j)^\pm \iff j \in D_\ell(\nu)^\pm. \tag{4}$$

We will base our proof on two observations;

Observation 5.4 *1. If* $\sum_{\ell=1}^{\infty} 2^{-\ell} \# M_\ell(j)^+ < 1 - \frac{1}{n^\eta}$ *then* $\omega(z_0, \tilde\gamma_{\kappa_j}; \Omega_2) \ge \frac{1}{n^{1+2\eta}}$.

2. If $\sum_{\ell=1}^{\infty} 2^{-\ell} \# M_\ell(j)^- < n^{-\eta}$ *then* $\omega(z_0, \tilde\gamma_{\kappa_j}; \Omega_2) \le \frac{1}{n^{1-2\eta}}$.

Proof. Given a sequence of numbers $i_1, \cdots, i_\nu \in \{1, 2, \cdots, m\}$ we define by $\Omega^{i_1 i_2 \cdots i_\nu}$ the domain obtained from Ω_1 by replacing $\gamma_{\kappa_{i_\ell}}$ with $\tilde\gamma_{\kappa_{i_\ell}}$ for all $1 \le \ell \le \nu$. We denote by Ω^{-j} the domain Ω_2 where $\tilde\gamma_{\kappa_j}$ is replaced with γ_{κ_j}.

By definition of Ω^{-j},

$$\omega(z_0, \gamma_{\kappa_j}; \Omega^{-j} \cap \Omega_1) = \omega(z_0, \gamma_{\kappa_j}; \Omega_1) - \sum_{\nu \ne j} \left(\omega(z_0, \gamma_{\kappa_j}; \Omega^{12\cdots\nu} \cap \Omega_1) - \omega(z_0, \gamma_{\kappa_j}; \Omega^{12\cdots(\nu+1)} \cap \Omega_1)\right).$$

We begin by noting that

$$\omega(z_0, \gamma_{\kappa_j}; \Omega^{12\cdots\nu} \cap \Omega_1) - \omega(z_0, \gamma_{\kappa_j}; \Omega^{12\cdots(\nu+1)} \cap \Omega_1) \le \omega(z_0, \gamma_{\kappa_j}; \Omega_1) - \omega(z_0, \gamma_{\kappa_j}; \Omega_1^{\nu+1} \cap \Omega_1),$$

since on $\partial\Omega^{12\cdots\nu}$ the left hand side is equal to zero and the right hand side is non-negative by inclusion, while on $[\phi(a_{\nu+1}), \phi(b_{\nu+1})]$, the inequality becomes

$$\omega(z_0, \gamma_{\kappa_j}; \Omega^{12\cdots\nu} \cap \Omega_1) \le \omega(z_0, \gamma_{\kappa_j}; \Omega_1),$$

which holds, again, due to inclusion. Using this inequality we get

$$\begin{aligned}\omega(z_0, \gamma_{\kappa_j}; \Omega^{-j} \cap \Omega_1) &\ge \omega(z_0, \gamma_{\kappa_j}; \Omega_1) - \sum_{\nu \ne j} \left(\omega(z_0, \gamma_{\kappa_j}; \Omega_1) - \omega(z_0, \gamma_{\kappa_j}; \Omega_1^{\nu+1} \cap \Omega_1)\right) \\ &\ge \omega(z_0, \gamma_{\kappa_j}; \Omega_1) \left(1 - \sum_{\ell=1}^{\infty} 2^{-\ell} \# M_\ell^+(j)\right) \ge \frac{1}{n}\left(1 - \left(1 - \frac{1}{n^\eta}\right)\right) = \frac{1}{n^{1+\eta}},\end{aligned}$$

as we assumed that $\sum_{\ell=1}^{\infty} 2^{-\ell} \# M_\ell^+(j) < 1 - \frac{1}{n^\eta}$.

It is left to show that if $\omega(z_0, \gamma_{\kappa_j}; \Omega^{-j} \cap \Omega_1) \ge \frac{1}{n^{1+\eta}}$ then $\omega(z_0, \tilde\gamma_{\kappa_j}; \Omega_2) \ge \frac{1}{n^{1+2\eta}}$. Let $\eta_j = \partial(\Omega_2 \cap \Omega_1) \cap \gamma_{\kappa_j}$ (see Figure 10). Intuitively, this is the boundary of Ω_1 between $\phi(a_j)$ and $\phi(b_j)$ where the domain is convex, therefore when replacing γ_{κ_j} with $\tilde{\gamma}_{\kappa_j}$ we end up making the domain larger. Note that the map $\zeta \mapsto \omega(\zeta, \gamma_{\kappa_j}; \Omega^{-j} \cap \Omega_1)$ is harmonic in $\Omega_2 \cap \Omega_1$. Then, following the maximum principle,

$$\omega(z_0, \gamma_{\kappa_j}; \Omega^{-j} \cap \Omega_1) \leq \omega(z_0, \left(\tilde{\gamma}_{\kappa_j} \cap \partial\left(\Omega_2 \cap \Omega_1\right)\right) \cup \eta_j; \Omega_2 \cap \Omega_1).$$

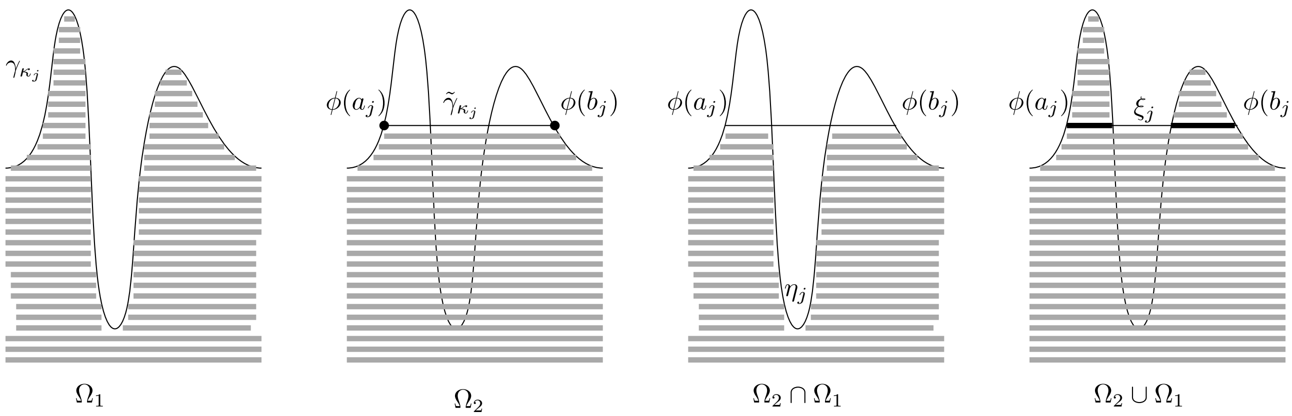


Figure 10: The sets Ω_1, Ω_2 their union and intersection.

On the other hand, by Beurling projection theorem, for every $\zeta \in \eta_j$ we have $\omega(\zeta, \tilde{\gamma}_{\kappa_j}; \Omega_2) \gtrsim c$ for some uniform constant $c \in (0,1)$, implying that

$$\begin{aligned}
\omega(z_0, \tilde{\gamma}_{\kappa_j}; \Omega_2) &= \int\limits_{\partial\Omega_2 \cap \Omega_1} \omega(\zeta, \tilde{\gamma}_{\kappa_j}; \Omega_2) d\omega(z_0, \zeta; \Omega_1 \cap \Omega_2) \\
&= \omega(z_0, \tilde{\gamma}_{\kappa_j} \cap \partial\left(\Omega_2 \cap \Omega_1\right); \Omega_1 \cap \Omega_2) + \int\limits_{\eta_j} \omega(\zeta, \tilde{\gamma}_{\kappa_j}; \Omega_2) d\omega(z_0, \zeta; \Omega_1 \cap \Omega_2) \\
&\geq c \cdot \omega(z_0, \left(\tilde{\gamma}_{\kappa_j} \cap \partial\left(\Omega_2 \cap \Omega_1\right)\right) \cup \eta_j; \Omega_2 \cap \Omega_1) \geq c \cdot \omega(z_0, \gamma_{\kappa_j}; \Omega^{-j} \cap \Omega_1) \geq c \cdot \frac{1}{n^{1+\eta}} \geq \frac{1}{n^{1+2\eta}},
\end{aligned}$$

if δ is numerically small enough, concluding the proof of the first part.

To prove the second part, we apply exactly the same argument, where intersections are replaces with unions and $\geq$ inequalities are replaces with $\leq$.

NOT TO INCLUDE IN PAPER:

By definition of Ω^{-j},

$$\omega(z_0, \gamma_{\kappa_j}; \Omega^{-j} \cup \Omega_1) = \omega(z_0, \gamma_{\kappa_j}; \Omega_1) + \sum_{\nu \neq j} \left(\omega(z_0, \gamma_{\kappa_j}; \Omega^{12\cdots(\nu+1)} \cup \Omega_1) - \omega(z_0, \gamma_{\kappa_j}; \Omega^{12\cdots\nu} \cup \Omega_1)\right).$$

We begin by noting that

$$\omega(z_0, \gamma_{\kappa_j}; \Omega^{12\cdots(\nu+1)} \cup \Omega_1) - \omega(z_0, \gamma_{\kappa_j}; \Omega^{12\cdots\nu} \cup \Omega_1) \geq \omega(z_0, \gamma_{\kappa_j}; \Omega_1^{\nu+1} \cup \Omega_1) - \omega(z_0, \gamma_{\kappa_j}; \Omega_1),$$

since on $\partial\Omega_1 \setminus \gamma_{\kappa_{\nu+1}}$ the left hand side is non-negative by inclusion, and the right hand side is zero, while on $\gamma_{\kappa_{\nu+1}}$, the inequality becomes

$$\omega(z_0, \gamma_{\kappa_j}; \Omega^{12\cdots\nu+1} \cup \Omega_1) \geq \omega(z_0, \gamma_{\kappa_j}; \Omega_1^{\nu+1} \cup \Omega_1),$$

which holds, again, due to inclusion. Using this inequality we get

$$\begin{aligned}\omega(z_0, \gamma_{\kappa_j}; \Omega^{-j} \cap \Omega_1) &\leq \omega(z_0, \gamma_{\kappa_j}; \Omega_1) + \sum_{\nu\neq j} \left(\omega(z_0, \gamma_{\kappa_j}; \Omega_1^{\nu+1} \cup \Omega_1) - \omega(z_0, \gamma_{\kappa_j}; \Omega_1)\right) \\ &\leq \omega(z_0, \gamma_{\kappa_j}; \Omega_1) \left(1 + \sum_{\ell=1}^{\infty} 2^{-\ell} \# M_\ell^-(j)\right) \leq \frac{1}{n}\left(1 + n^{-\eta}\right) = \frac{2}{n^{1-\eta}},\end{aligned}$$

as we assumed that $\sum_{\ell=1}^{\infty} 2^{-\ell} \# M_\ell^+(j) < n^{-\eta}$.

It is left to show that if $\omega(z_0, \gamma_{\kappa_j}; \Omega^{-j} \cup \Omega_1) \leq \frac{1}{n^{1-\eta}}$ then $\omega(z_0, \tilde{\gamma}_{\kappa_j}; \Omega_2) \leq \frac{1}{n^{1-2\eta}}$. Let $\xi_j = \tilde{\gamma}_{\kappa_j} \cap \Omega_1$. Note that the map $\zeta \mapsto \omega(\zeta, \left(\gamma_{\kappa_j} \cup \tilde{\gamma}_{\kappa_j}\right) \setminus (\Omega_1 \cup \Omega_2); \Omega_2 \cup \Omega_1)$ is harmonic in $\Omega^{-j} \cup \Omega_1$. Then, following the maximum principle,

$$\omega(z_0, \gamma_{\kappa_j}; \Omega^{-j} \cup \Omega_1) \geq \omega(z_0, \left(\gamma_{\kappa_j} \cup \tilde{\gamma}_{\kappa_j}\right) \setminus (\Omega_1 \cup \Omega_2); \Omega_2 \cup \Omega_1).$$

On the other hand, by Beurling projection theorem, for every $\zeta \in \xi_j$ we have

$$\omega(\zeta, \left(\gamma_{\kappa_j} \cup \tilde{\gamma}_{\kappa_j}\right) \setminus (\Omega_1 \cup \Omega_2); \Omega_2 \cup \Omega_1) = \omega(\zeta, \tilde{\gamma}_{\kappa_j} \setminus \Omega_1; \Omega_2 \cup \Omega_1) \geq c$$

for some uniform constant $c \in (0, 1)$, implying that

$$\begin{aligned}\omega(z_0, \left(\gamma_{\kappa_j} \cup \tilde{\gamma}_{\kappa_j}\right) \setminus (\Omega_1 \cup \Omega_2); \Omega_2 \cup \Omega_1) &= \int_{\partial\Omega_2} \omega(\zeta, \tilde{\gamma}_{\kappa_j} \setminus \Omega_1; \Omega_2 \cup \Omega_1) d\omega(z_0, \zeta; \Omega_2) \\ &\geq \omega(z_0, \tilde{\gamma}_{\kappa_j} \setminus \Omega_1; \Omega_2) + \int_{\xi_j} \omega(\zeta, \tilde{\gamma}_{\kappa_j} \setminus \Omega_1; \Omega_2 \cup \Omega_1) d\omega(z_0, \zeta; \Omega_2) \\ &\geq c \cdot \omega(z_0, \tilde{\gamma}_{\kappa_j}; \Omega_2).\end{aligned}$$

Combining these bounds together we see that

$$\omega(z_0, \tilde{\gamma}_{\kappa_j}; \Omega_2) \leq \frac{1}{c}\omega(z_0, \left(\gamma_{\kappa_j} \cup \tilde{\gamma}_{\kappa_j}\right) \setminus (\Omega_1 \cup \Omega_2); \Omega_2 \cup \Omega_1) \leq \frac{1}{c}\omega(z_0, \gamma_{\kappa_j}; \Omega^{-j} \cup \Omega_1) \leq \frac{1}{c} \cdot \frac{2}{n^{1-\eta}} \leq \frac{1}{n^{1-2\eta}}$$

if δ is numerically small enough, concluding the proof of the second part. ☐

Observation 5.5

$$\sum_{\ell=1}^{\infty} 2^{-\ell} \# D_\ell^+(j) \leq \frac{4}{n^\eta} \qquad , \qquad \sum_{\ell=1}^{\infty} 2^{-\ell} \# D_\ell^-(j) \leq 2.$$

Proof. Recall that by definition of the collections $\left\{D_\ell^+(j)\right\}$, if $\nu \in D_\ell^+(j)$, then

$$1 - \frac{\omega(z_0, \gamma_{\kappa_\nu}; \Omega_1^j \cap \Omega_1)}{\omega(z_0, \gamma_{\kappa_\nu}; \Omega_1)} \in \left[2^{-\ell-1}, 2^{-\ell}\right)$$

implying that

$$\omega(z_0, \gamma_{\kappa_\nu}; \Omega_1) \cdot 2^{-\ell-1} \leq \omega(z_0, \gamma_{\kappa_\nu}; \Omega_1) - \omega(z_0, \gamma_{\kappa_\nu}; \Omega_1^j \cap \Omega_1).$$

In addition, the collections $\{D_\ell(j)\}$ are disjoint. We see that for $\eta_j := \partial(\Omega_2 \cap \Omega_1) \cap \gamma_{\kappa_j}$

$$
\begin{aligned}
\sum_{\ell=1}^{\infty} 2^{-\ell} \# D_\ell^+(j) &= \sum_{\ell=1}^{\infty} \frac{\omega(z_0,\gamma_{\kappa_j};\Omega_1)}{\omega(z_0,\gamma_{\kappa_j};\Omega_1)} \cdot 2^{-\ell} \# D_\ell^+(j) = \frac{2}{\omega(z_0,\gamma_{\kappa_j};\Omega_1)} \sum_{\ell=1}^{\infty} \sum_{\nu \in D_\ell^+(j)} \omega(z_0,\gamma_{\kappa_j};\Omega_1) \cdot 2^{-\ell-1} \\
&\le \frac{2}{\omega(z_0,\gamma_{\kappa_j};\Omega_1)} \sum_{\ell=1}^{\infty} \sum_{\nu \in D_\ell^+(j)} \Big(\omega(z_0,\gamma_{\kappa_\nu};\Omega_1) - \omega(z_0,\gamma_{\kappa_\nu};\Omega_1^j \cap \Omega_1) \Big) \\
&= \frac{2}{\omega(z_0,\gamma_{\kappa_j};\Omega_1)} \sum_{\ell=1}^{\infty} \sum_{\nu \in D_\ell^+(j)} \int_{\partial(\Omega_1^j \cap \Omega_1)} \omega(\zeta,\gamma_{\kappa_\nu};\Omega_1) - \omega(\zeta,\gamma_{\kappa_\nu};\Omega_1^j \cap \Omega_1) d\omega(z_0,\zeta;\Omega_1^j \cap \Omega_1) \\
&= \frac{2}{\omega(z_0,\gamma_{\kappa_j};\Omega_1)} \sum_{\ell=1}^{\infty} \sum_{\nu \in D_\ell^+(j)} \int_{\eta_j} \omega(\zeta,\gamma_{\kappa_\nu};\Omega_1) d\omega(z_0,\zeta;\Omega_1^j \cap \Omega_1) \\
&\overset{\substack{\text{disjointness}\\ \text{of } D_\ell^+(j)}}{\le} \frac{2}{\omega(z_0,\gamma_{\kappa_j};\Omega_1)} \sum_{\nu=1}^{m} \int_{\eta_j} \omega(\zeta,\gamma_{\kappa_\nu};\Omega_1) d\omega(z_0,\zeta;\Omega_1^j \cap \Omega_1) \\
&\overset{\substack{\text{disjointness}\\ \text{of } \gamma_{\kappa_\nu}}}{=} \frac{2}{\omega(z_0,\gamma_{\kappa_j};\Omega_1)} \int_{\eta_j} \omega\left(\zeta, \bigcup_{\nu=1}^{m} \gamma_{\kappa_\nu};\Omega_1\right) d\omega(z_0,\zeta;\Omega_1^j \cap \Omega_1) \\
&\le \frac{2}{\omega(z_0,\gamma_{\kappa_j};\Omega_1)} \int_{\eta_j} 1 d\omega(z_0,\zeta;\Omega_1^j \cap \Omega_1) = \frac{2\omega(z_0,\eta_j;\Omega_1^j \cap \Omega_1)}{\omega(z_0,\gamma_{\kappa_j};\Omega_1)} \\
&\le \frac{2\omega(z_0,[\phi(a_j),\phi(b_j)];\Omega_1^j)}{\omega(z_0,\gamma_{\kappa_j};\Omega_1)} \le n \cdot \frac{4}{n^{1+\eta}} = \frac{4}{n^\eta},
\end{aligned}
$$

The case of $D_\ell(j)^-$ is shown similarly. This concludes our proof.

NOT TO INCLUDE IN PAPER:

Recall that by definition of the collections $\{D_\ell^-(j)\}$, if $\nu \in D_\ell^-(j)$, then

$$\frac{\omega(z_0,\gamma_{\kappa_\nu};\Omega_1^j \cup \Omega_1)}{\omega(z_0,\gamma_{\kappa_\nu};\Omega_1)} - 1 \in \left[2^{-\ell-1}, 2^{-\ell}\right)$$

implying that

$$\omega(z_0,\gamma_{\kappa_\nu};\Omega_1) \cdot 2^{-\ell-1} \le \omega(z_0,\gamma_{\kappa_\nu};\Omega_1^j \cup \Omega_1) - \omega(z_0,\gamma_{\kappa_\nu};\Omega_1).$$

In addition, the collections $\{D_\ell(j)\}$ are disjoint. We see that

$$
\begin{aligned}
\sum_{\ell=1}^{\infty} 2^{-\ell} \# D_\ell^-(j) &= \sum_{\ell=1}^{\infty} \frac{\omega(z_0,\gamma_{\kappa_j};\Omega_1)}{\omega(z_0,\gamma_{\kappa_j};\Omega_1)} \cdot 2^{-\ell} \# D_\ell^-(j) = \frac{2}{\omega(z_0,\gamma_{\kappa_j};\Omega_1)} \sum_{\ell=1}^{\infty} \sum_{\nu \in D_\ell^-(j)} \omega(z_0,\gamma_{\kappa_j};\Omega_1) \cdot 2^{-\ell-1} \\
&\le \frac{2}{\omega(z_0,\gamma_{\kappa_j};\Omega_1)} \sum_{\ell=1}^{\infty} \sum_{\nu \in D_\ell^-(j)} \Big(\omega(z_0,\gamma_{\kappa_\nu};\Omega_1^j \cup \Omega_1) - \omega(z_0,\gamma_{\kappa_\nu};\Omega_1) \Big) \\
&= \frac{2}{\omega(z_0,\gamma_{\kappa_j};\Omega_1)} \sum_{\ell=1}^{\infty} \sum_{\nu \in D_\ell^-(j)} \int_{\eta_j} \omega(\zeta,\gamma_{\kappa_\nu};\Omega_1) d\omega(z_0,\zeta;\Omega_1) \\
&\overset{\substack{\text{disjointness}\\ \text{of } D_\ell^-(j)}}{\le} \frac{2}{\omega(z_0,\gamma_{\kappa_j};\Omega_1)} \sum_{\nu=1}^{m} \int_{\eta_j} \omega(\zeta,\gamma_{\kappa_\nu};\Omega_1) d\omega(z_0,\zeta;\Omega_1) \\
&\overset{\substack{\text{disjointness}\\ \text{of } \gamma_{\kappa_\nu}}}{=} \frac{2}{\omega(z_0,\gamma_{\kappa_j};\Omega_1)} \int_{\eta_j} \omega\left(\zeta, \bigcup_{\nu=1}^{m} \gamma_{\kappa_\nu};\Omega_1\right) d\omega(z_0,\zeta;\Omega_1) \\
&\le \frac{2}{\omega(z_0,\gamma_{\kappa_j};\Omega_1)} \int_{\eta_j} 1 d\omega(z_0,\zeta;\Omega_1) = \frac{2\omega(z_0,\eta_j;\Omega_1)}{\omega(z_0,\gamma_{\kappa_j};\Omega_1)} \le 2.
\end{aligned}
$$

□

We are now ready to prove Lemma 5.3. In fact, this will now be a simple computation, based on the relationship between $M_\ell^{\pm}(j)$ and $D_\ell^{\pm}(\nu)$ introduced above, (4), and the two observations-

$$
\begin{aligned}
m\cdot\frac{4}{n^\eta} &= \sum_{\nu=1}^m \frac{4}{n^\eta} \overset{\text{By Obs 5.5}}{\geq} \sum_{\nu=1}^m\sum_{\ell=1}^\infty 2^{-\ell}\#D_\ell^+(\nu) = \sum_{\nu=1}^m\sum_{\ell=1}^\infty 2^{-\ell}\sum_{j\in D_\ell^+(\nu)} 1 \\
&\overset{\text{Eq. (4)}}{=} \sum_{\ell=1}^\infty 2^{-\ell}\sum_{j=1}^m\sum_{\nu\in M_\ell^+(j)} 1 = \sum_{j=1}^m\sum_{\ell=1}^\infty 2^{-\ell}\#M_\ell^+(j) \geq \sum_{j,\omega(z_0,\tilde\gamma_{\kappa_j};\Omega_2)<\frac{1}{n^{1+2\eta}}}\sum_{\ell=1}^\infty 2^{-\ell}\#M_\ell^+(j) \\
&\overset{\text{By Obs 5.4}}{\geq} \#\left\{j,\omega(z_0,\tilde\gamma_{\kappa_j};\Omega_2)<\frac{1}{n^{1+2\eta}}\right\}\cdot\left(1-\frac{1}{n^\eta}\right) \\
&\Rightarrow \#\left\{j,\omega(z_0,\tilde\gamma_{\kappa_j};\Omega_2)<\frac{1}{n^{1+2\eta}}\right\} \leq \frac{4m}{n^\eta\left(1-\frac{1}{n^\eta}\right)} < \frac{m}{4},
\end{aligned}
$$

as long as $n^\eta > 17$.

A similar computation shows that

$$\#\left\{j,\omega(z_0,\tilde\gamma_{\kappa_j};\Omega_2)>\frac{1}{n^{1-2\eta}}\right\} \leq \frac{2m}{n^\eta} < \frac{m}{4}$$

concluding the proof.

NOT TO INCLUDE IN PAPER:

$$
\begin{aligned}
2m &= \sum_{\nu=1}^m 2 \overset{\text{By Obs 5.5}}{\geq} \sum_{\nu=1}^m\sum_{\ell=1}^\infty 2^{-\ell}\#D_\ell^-(\nu) = \sum_{\nu=1}^m\sum_{\ell=1}^\infty 2^{-\ell}\sum_{j\in D_\ell^-(\nu)} 1 \\
&\overset{\text{Eq. (4)}}{=} \sum_{\ell=1}^\infty 2^{-\ell}\sum_{j=1}^m\sum_{\nu\in M_\ell^-(j)} 1 = \sum_{j=1}^m\sum_{\ell=1}^\infty 2^{-\ell}\#M_\ell^-(j) \geq \sum_{j,\omega(z_0,\tilde\gamma_{\kappa_j};\Omega_2)>\frac{1}{n^{1-2\eta}}}\sum_{\ell=1}^\infty 2^{-\ell}\#M_\ell^-(j) \\
&\overset{\text{By Obs 5.4}}{\geq} \#\left\{j,\omega(z_0,\tilde\gamma_{\kappa_j};\Omega_2)>\frac{1}{n^{1-2\eta}}\right\}\cdot n^\eta \\
&\Rightarrow \#\left\{j,\omega(z_0,\tilde\gamma_{\kappa_j};\Omega_2)>\frac{1}{n^{1-2\eta}}\right\} \leq \frac{2m}{n^\eta} < \frac{m}{4},
\end{aligned}
$$

□

Lemma 5.6 *For every j, if $\omega(z_0,\tilde\gamma_{\kappa_j};\Omega_2) > \frac{1}{n^{1+2\eta}}$ then $\omega(z_0,[\phi(a_j),\phi(b_j)];\Omega_2) > \frac{1}{n^{1+4\eta}}$, as long as η is numerically small enough.*

Proof. We shall use the notion of extremal length combined with Whitney cubes. Fix j and let $I_j := \phi_2^{-1}\left(\tilde\gamma_{\kappa_j}\right)$, where $\phi_2:\mathbb{D}\to\Omega_2$ is a Riemann map. Let C_j be the Whitney cube generated by the arc I_j, and let $T_j := \phi_2(C_j)$.

Fix a curve σ connecting z_0 with $\partial\Omega_2\setminus\tilde\gamma_{\kappa_j}$ and let Γ denote the collection of all curves in Ω_2 connecting σ with $\tilde\gamma_{\kappa_j}$. We write $\Gamma = \Gamma_1+\Gamma_2$ where

$$\begin{cases}\Gamma_1 := \{\gamma\cap(\Omega_2\setminus T_j),\gamma\in\Gamma\} \\ \Gamma_2 := \{\gamma\cap T_j,\gamma\in\Gamma\}\end{cases}.$$

Following the parallel rule (see, e.g., [29, p.136])

$$\frac{1}{\lambda_{\Omega_2}(\Gamma)} \geq \frac{1}{\lambda_{(\Omega_2 \setminus T_j)}(\Gamma_1)} + \frac{1}{\lambda_{T_j}(\Gamma_2)}.$$

Let us bound each of these terms independently. For the first quantity, we use the extension rule, with $\Omega' = \Omega_2$ and Γ' the collection of all curves in Ω_2 connecting σ with $\tilde{\gamma}_{\kappa_j}$. Then, following [29, Theorem 5.2 p. 145],

$$\lambda_{(\Omega_2 \setminus T_j)}(\Gamma_1) \leq \lambda_{\Omega_2}(\Gamma') \leq \frac{1}{\pi} \log\left(\frac{8}{\pi\omega(z_0, \tilde{\gamma}_{\kappa_j}; \Omega_2)}\right) < \frac{1}{\pi} \log\left(\frac{8}{\pi \cdot \frac{1}{n^{1+2\eta}}}\right).$$

Next, because the extremal length is conformal invariant, and

$$\frac{diam\left([\phi(a_j), \phi(b_j)]\right)}{diam(\tilde{\gamma}_{\kappa_j})} \geq \frac{\frac{1}{n^\eta} diam(\tilde{\gamma}_{\kappa_j})}{diam(\tilde{\gamma}_{\kappa_j})} = \frac{1}{n^\eta},$$

there exists some uniform $C > 1$ so that

$$\lambda_{T_j}(\Gamma_2) \leq \frac{1}{\pi} \log\left(\frac{8}{\pi \cdot C \frac{1}{n^\eta}}\right).$$

Combining these estimates we see that

$$\frac{1}{\lambda_{\Omega_2}(\Gamma)} \geq \frac{\pi}{\log\left(\frac{8}{\pi \cdot \frac{1}{n^{1+2\eta}}}\right)} + \frac{\pi}{\log\left(\frac{8}{\pi \cdot C \frac{1}{n^\eta}}\right)} = \frac{\pi}{\log(n)} \left(\frac{1}{1+2\eta} + \frac{1}{\eta}\right)(1 - o(1))$$

$$\Rightarrow \lambda_{\Omega_2}(\Gamma) \leq \frac{1+3\eta}{\pi} \log(n)\left(1 + o(1)\right),$$

since for $a, b > 0$ we have $\frac{1}{\frac{1}{a}+\frac{1}{b}} \leq a + b$.

Lastly, note that this holds for every σ implying that

$$\omega(z_0, [\phi(a_j), \phi(b_j)]; \Omega_2) \geq \frac{1}{\pi} e^{-\pi\lambda(z_0, [\phi(a_j), \phi(b_j)])} \geq \frac{1}{\pi} e^{-\pi \frac{1+3\eta}{\pi} \log(n)(1+o(1))} \geq \frac{1}{n^{1+4\eta}},$$

as long as η is numerically small enough. □

We conclude that for at least half of the elements in the collection $\left\{\gamma_{\kappa_j}\right\}$ we have

$$\frac{1}{n^{1+4\eta}} \leq \omega\left(z_0, [\phi(a_j), \phi(b_j)]; \Omega_2\right) \leq \frac{1}{n^{1-2\eta}}. \tag{5}$$

5.2.3 Step 3: The disks

We cover $\partial\Omega_2 \setminus \biguplus_{j \text{ satisfies } (5)} [\phi(a_j), \phi(b_j)]$ with tangential disks of radius $\frac{1}{n^4}$. Denote by P the polygon whose boundary is the union of lines connecting the tangential points of every two consecutive disks together with $\biguplus_{j \text{ satisfies } (5)} [\phi(a_j), \phi(b_j)]$.

Lemma 5.7 *For every j satisfying* (5)

$$\frac{1}{2n^{1+4\eta}} \leq \omega\left(z_0, [\phi(a_j), \phi(b_j)]; P\right) \leq \frac{2}{n^{1-2\eta}}.$$

Proof. Note that $P \cap \Omega_2 \subseteq P \subseteq P \cup \Omega_2$, then

$$\begin{aligned}\omega(z_0, [\phi(a_j), \phi(b_j)]\,; P) &= \int\limits_{\partial(P\cap\Omega_2)} \omega(\zeta, [\phi(a_j), \phi(b_j)]\,; P) d\omega(z_0, \zeta; P\cap\Omega_2) \\ &= \omega(z_0, [\phi(a_j), \phi(b_j)]\,; P\cap\Omega_2) + \int\limits_{\partial P\cap\Omega_2} \omega(\zeta, [\phi(a_j), \phi(b_j)]\,; P) d\omega(z_0, \zeta; P\cap\Omega_2) \\ &\leq \frac{1}{n^{1-2\eta}} + \int\limits_{\partial P\cap\Omega_2} \frac{1}{n^2} d\omega(z_0, \zeta; P\cap\Omega_2) \leq \frac{1}{n^{1-2\eta}} + \frac{1}{n^2} \leq \frac{2}{n^{1-2\eta}}\end{aligned}$$

since $d_{\mathcal{H}}(\partial P, \partial\Omega_2) \leq \frac{1}{n^2}$ and $[\phi(a_j), \phi(b_j)] \notin \partial P \cap \Omega_2$, then following Beurling for every $\zeta \in \partial P \cap \Omega_2$ we have

$$\omega(\zeta, [\phi(a_j), \phi(b_j)]\,; P) \leq \sqrt{\frac{1}{n^4}} = \frac{1}{n^2}.$$

A similar computation with $P \cup \Omega_2$ shows that $\omega(z_0, [\phi(a_j), \phi(b_j)]\,; P) \geq \frac{1}{2n^{1+4\eta}}$, concluding the proof.

NOT TO INCLUDE IN PAPER:

$$\begin{aligned}\omega(z_0, [\phi(a_j), \phi(b_j)]\,; P\cup\Omega_2) &= \int\limits_{\partial(P\cup\Omega_2)} \omega(\zeta, [\phi(a_j), \phi(b_j)]\,; P) d\omega(z_0, \zeta; P\cup\Omega_2) \\ &= \omega(z_0, [\phi(a_j), \phi(b_j)]\,; P) + \int\limits_{\partial P\setminus\Omega_2} \omega(\zeta, [\phi(a_j), \phi(b_j)]\,; P) d\omega(z_0, \zeta; P\cup\Omega_2) \\ &\leq \omega(z_0, [\phi(a_j), \phi(b_j)]\,; P) + \int\limits_{\partial P\setminus\Omega_2} \frac{1}{n^2} d\omega(z_0, \zeta; P\cap\Omega_2),\end{aligned}$$

implying that

$$\begin{aligned}\omega(z_0, [\phi(a_j), \phi(b_j)]\,; P) &\geq \omega(z_0, [\phi(a_j), \phi(b_j)]\,; P\cup\Omega_2) - \frac{1}{n^2} \geq \omega(z_0, [\phi(a_j), \phi(b_j)]\,; \Omega_2) - \frac{1}{n^2} \\ &\geq \frac{1}{n^{1+4\eta}} - \frac{1}{n^4} \geq \frac{1}{2n^{1+4\eta}}.\end{aligned}$$

□

For every j satisfying (5), we cover the segment $[\phi(a_j), \phi(b_j)]$ with tangential disks of doubling radius starting from the two disks of radius $\frac{1}{n^4}$ at the endpoint of the interval, working our way inside. We stop if the radius exceeds $diam(\gamma_{\kappa_j}) \cdot n^{-\eta}$ and cover the rest with disks of the same radius. The number of disks used to cover such a segment is bounded by

$$\#\left\{disks\right\} \lesssim \log(n) + \frac{diam(\gamma_{\kappa_j})}{diam(\gamma_{\kappa_j})n^{-\eta}} \lesssim \log(n) + n^{\eta} \leq 2n^{\eta},$$

for n large enough.

In particular, if j satisfies (5), one of the disks in the middle of the segment has diameter at least $diam(\gamma_{\kappa_j}) \cdot n^{-2\eta}$ and by inclusion harmonic measure (with respect to the polygon, P) at most $\frac{1}{n^{1-2\eta}}$, while by the pigeon-hole principle, at least one of the disks has harmonic measure

$$\frac{\omega(z_0, [\phi(a_j), \phi(b_j)]\,; P)}{\#\left\{disks\right\}} \geq \frac{\frac{1}{2n^{1+4\eta}}}{2n^{\eta}} = \frac{1}{4n^{1+5\eta}},$$

and diameter less than the diameter of γ_{κ_j}.

Claim 5.8 *For every disk, D, in the collection described above,* $\#\left\{B, B\cap \frac{3}{2}D \neq \emptyset\right\} = 3$.

Note that this implies that the only disks in the intersection are the ones tangential to D, which is needed to define the dynamics on this iterated functions system. In addition, it implies that the segments defining ∂P are pairwise disjoint, that is P is simply connected.

Proof. Let D, D' be two disks in the collection and assume without loss of generality that the radius of D, $r(D)$, is no smaller that the radius of D'. If $D'\cap \frac{3}{2}D \neq \emptyset$ then there exist $\zeta \in D, \zeta' \in D'$ so that

$$|\zeta - \zeta'| \leq diam(D') + diam\left(\frac{3}{2}D\right) = 5r(D),$$

and for some $z, z' \in \left(1 - \frac{1}{n}\right)\mathbb{T}$, $\zeta = \phi(z),\ \zeta' = \phi(z')$.

We will look at two cases:

Case 1: $r(D) = \frac{1}{n^4}$. Then using distortion arguments (see, e.g., [48, Cor 1.5 p.10]),

$$|\zeta - \zeta'| \geq \frac{1}{4}\tanh(\rho(z,z'))(1-|z|^2)\,|\phi'(z)| \geq \frac{1}{64}\rho(z,z')(1-|z|)^2 = \frac{1}{64\cdot n^2}\rho(z,z')$$

implying that

$$\rho(z,z') \leq 64\cdot n^2\,|\zeta-\zeta'| \leq 320n^2\cdot r(D) = \cdot\frac{320n^2}{n^4} = \frac{320}{n^2}$$

In other words, as long as n is numerically large enough, z, z' either belong to the same curve, γ_k, or to neighbouring ones. Recall that $\log\phi'$ is a Bloch function, implying that

$$|arg\,(\phi'(z)) - arg\,(\phi'(z'))| \leq |\log\,(\phi'(z)) - \log\,(\phi'(z'))| \leq 6\rho(z,z') \lesssim \frac{1}{n^2}.$$

In particular, the only possibility for such intersection is if D, D' are tangential, because the disks have the same radius.

Case 2: If the radius of D is bigger than $\frac{1}{n^4}$, then there exists j satisfying (5) so that D sits on a segment $[\phi(a_j), \phi(b_j)]$ with P-harmonic measure at least $\frac{1}{4n^{1+5\eta}}$ and diameter bounded by $diam(\gamma_{\kappa_j})$.

Since $\log\phi_0'$ is a Bloch function, for every $z\in[z_k, z_{k+1}]$

$$diam(\gamma_k) = \sup_{\zeta,\eta\in[z_k,z_{k+1}]}|\phi(\zeta)-\phi(\eta)| \sim \int_{z_k}^{z_{k+1}}|\phi'|d|\zeta| \sim |z_k - z_{k+1}|\,|\phi'(z)| \sim (1-|z|)\,|\phi'(z)|\,.$$

In particular,

$$|\phi(z)-\phi(z')| \gtrsim \rho(z,z')\,(1-|z|)\,|\phi'(z)| \gtrsim \rho(z,z')\cdot diam(\gamma_k).$$

We conclude that since the disks D are defined so that $r(D)\leq diam(\gamma_k)\cdot n^{-\eta}$,

$$\rho(z,z') \lesssim \frac{|\phi(z)-\phi(z')|}{diam(\gamma_k)} \leq \frac{5r(D)}{diam(\gamma_k)} \lesssim \frac{1}{n^\eta}.$$

If the disks are not tangential, then the part of $\partial\Omega_1$ between D and D' cannot be a graph, in particular, the argument has to shift by at least $\pm\frac{\pi}{2}$, that is

$$\frac{\pi}{2} \leq |arg\,(\phi'(z)) - arg\,(\phi'(z'))| \leq 6\rho(z,z') \lesssim \frac{1}{n^\eta} < 1,$$

if n is numerically large enough. As before, the only possibility for such intersection is if D, D' are tangential, because the ratio between their radii is in the set $\left\{1, \frac{1}{2}, 2\right\}$. $\square$

Theorem 5.9 *Let $\Omega_0 \subset \mathbb{C}$ be a simply connected domain, and fix $\alpha > 0$, $\eta > 0$ small enough (depending on α), and δ small enough (depending on α and η). Then there exists a polygon P and collection of tangential disks $\{D\}_{D\in\mathcal{P}}$ covering ∂P with $\#\left\{D \cap \frac{3}{2}D' \neq \emptyset\right\} = 3$ so that*

$$N_P^+(\delta, \alpha, \eta\,(5\alpha + 12)) \gtrsim N_{\Omega_0}^+(\delta, \alpha, \eta).$$

Proof. Given δ, η we define $n := \lceil \delta^{-\alpha-2\eta} \rceil \in \left[\delta^{-\alpha-2\eta}, \delta^{-\alpha-2\eta} + 1\right]$. Let $\{B_j\}_{j=1}^{N_{\Omega_0}(\delta,\alpha;\eta)}$ be the maximal collection of disks defining $N_{\Omega_0}^+(\delta, \alpha; \eta)$. Following Carleson's lemma (see, e.g., [29, Lemma 2.5 p.277]) for every j there exists a continuum $\beta_j \subset 2B_j \cap \partial\Omega_0$ with harmonic measure exactly $\delta^{\alpha+2\eta}$. By excluding a fixed linear portion of the disks in the collection $\{B_j\}$, we may assume that $\{2B_j\}$ are pairwise disjoint. Define $w_j := \left(1 - \frac{1}{n}\right)\zeta_j$, where ζ_j is the centre of the arc $\phi_0^{-1}(\beta_j) \subset \mathbb{T}$. Let $\Omega_1 = \phi\,(\mathbb{D})$ be as in the statement of Theorem 5.2 and note that as $w_j \in \partial\Omega_1$ therefor there exists κ_j so that $w_j \in \left[z_{\kappa_j}, z_{\kappa_{j+1}}\right]$. We will show that $diam(\gamma_{\kappa_j}) \lesssim diam(\beta_j)$.

$$\begin{aligned} diam(\gamma_{\kappa_j}) &:= \sup_{\zeta,\eta\in\left[z_{\kappa_j}, z_{\kappa_{j+1}}\right]} |\phi(\zeta) - \phi(\eta)| = \sup_{\zeta,\eta\in\left[z_{\kappa_j}, z_{\kappa_{j+1}}\right]} |\phi'(\xi_{\zeta,\eta})|\,|\zeta - \eta| \lesssim |\phi'(w_j)|\left|z_{\kappa_j} - z_{\kappa_{j+1}}\right| \\ &\sim |\phi_0'(w_j)|\left|z_{\kappa_j} - z_{\kappa_{j+1}}\right| \sim \frac{dist(\phi_0(w_j), \partial\Omega_0)}{1 - |w_j|^2}\left|z_{\kappa_j} - z_{\kappa_{j+1}}\right| \lesssim diam(\beta_j), \end{aligned}$$

by Observation 3.3 part 1.

Next, because β_j are disjoint, and have harmonic measure $\delta^{\alpha+2\eta}$, then the arcs $\phi_0^{-1}(\beta_j)$ are disjoint and have length $\delta^{\alpha+2\eta}$. In particular, for every k fixed, the number of such continuums intersecting γ_k is bounded by

$$\frac{\lambda_1(\left[z_{\kappa_j}, z_{\kappa_j+1}\right])}{\lambda_1(\phi_0^{-1}(\beta_j))} = \frac{\omega(z_0, \gamma_{\kappa_j}; \Omega_1)}{\omega(z_0, \beta_j; \Omega_0)} \leq \frac{\frac{1}{n}}{\delta^{\alpha+2\eta}} \leq 2.$$

By again excluding a linear portion of the disks left in the collection, we may assume the correspondence $j \mapsto \kappa_j$ is one to one. Let $m \in \mathbb{N}$ be the maximal number of disks in the collection satisfying that $j \mapsto \kappa_j$ is a one to one map. Because we only excluded a liner portion of the disks, $m \sim N_{\Omega_0}^+(\delta, \alpha; \eta)$.

We now apply Theorem 5.2 to the collection $\left\{\gamma_{\kappa_j}\right\}$ to get a polygon P whose boundary is covered by a collection of disjoint disks satisfying $\#\left\{D \cap \frac{3}{2}D' \neq \emptyset\right\} = 3$ and property (2a). In particular, every such disk is counted in $N_P^+(\delta, \alpha, \eta\,(5\alpha + 12))$ and therefore we get that $N_P^+(\delta, \alpha, \eta\,(5\alpha + 12)) \gtrsim m \sim N_{\Omega_0}^+(\delta, \alpha, \eta)$ concluding the proof. $\square$

As for the Minkowski distortion spectrum, we will show that Ω_0 can be approximated by a polygon, P, with a large lower bound on $\#\Gamma_P\,(a', r)$.

Theorem 5.10 *Let $\Omega_0 \subset \mathbb{C}$ be a simply connected domain, and fix $a > 0$, η small enough (depending on a), and $\varepsilon \in (0,1)$ small enough (depending on a and η). Then there exist a polygon P and collection of tangential disks $\{D\}_{D\in\mathcal{P}}$ covering ∂P with $\#\left\{D \cap \frac{3}{2}D' \neq \emptyset\right\} = 3$ so that*

$$\#\Gamma_{\Omega_0}(a,r) \lesssim \#\Gamma_P(a',r')$$

where $(1-r') = (1-r)^{(1+5\eta)}$ and $a' = a\,(1+O(\eta))$.

Proof. Let $n = \left\lceil \frac{1}{1-r} \right\rceil \in \left[\frac{1}{1-r}, \frac{1}{1-r}+1\right]$, and $\left\{\gamma_{\kappa_j}\right\}_{j=1}^m$ be the maximal number of arcs so that for every j there exists $\gamma \in \Gamma(a',r)$ so that $\left(1-\frac{1}{n}\right)A(\gamma)\cap\gamma_{\kappa_j} \neq \emptyset$. Since the intersection if non-empty and they both sit at the same distance from the boundary, by the fact that $\log \phi'$ is Bloch, they have comparable length, implying that this correspondence is finite to finite, and be excluding a linear portion of the curves in $\Gamma_{\Omega_0}(a,r)$ we are left with a one to one correspondence.

Then we count curves γ_{κ_j} with $diam(\gamma_{\kappa_j}) \geq (1-r)^{1-a'}$. Recall that on $\left(1-\frac{1}{n}\right)A(\gamma)$ we have $|\phi_0'| \gtrsim n^a$ and so if $\zeta_0 \in \left(1-\frac{1}{n}\right)A(\gamma)\cap\gamma_{\kappa_j}$ then

$$\begin{aligned} length(\gamma_{\kappa_j}) &= \int_{z_{\kappa_j}}^{z_{\kappa_j+1}} |\phi'(\zeta)||d|\zeta| \sim \left|z_{\kappa_j} - z_{\kappa_j+1}\right|\left|\phi'(z_{\kappa_j})\right| \sim \left|z_{\kappa_j} - z_{\kappa_j+1}\right||\phi_0'(\zeta_0)| \\ &\gtrsim \frac{n^a}{n} \sim (1-r)^{1-a}. \end{aligned}$$

In particular, $diam(\gamma_{\kappa_j}) > (1-r)^{1-a+\eta}$ for r close enough to 1.

We now apply Theorem 5.2 to the collection $\left\{\gamma_{\kappa_j}\right\}$ to get a polygon P whose boundary is covered by a collection of disjoint disks satisfying $\#\left\{D \cap \frac{3}{2}D' \neq \emptyset\right\} = 3$ and property (2a). In particular, by defining $r' = (1-r)^{1+5\eta}$, every such curve is counted in $\#\Gamma_P(a',r')$ for $a' = a\,(1+O(\eta))$, and therefore we get that $\#\Gamma_P(a',r') \gtrsim \#\Gamma_{\Omega_0}(a,r)$ concluding the proof. $\square$

5.3 Iterated Function Systems

Lemma 5.11 *Let P be a symmetric polygon and assume that there is a collection of tangential disks $\{D_j\}$ satisfying that for every disk, D,*

1. *For every k, $\#\left\{j, D_j \cap \frac{3}{2}D_k \neq \emptyset\right\} = 3$.*
2. *$\partial P \cap D_j$ is a line segment.*
3. *The disks intersecting the real axis and their neighbours have radius $\frac{1}{n^2}$.*

Then there exists an iterated functions system , Ω_F, defined by a collection of open sets $\{U_j\}$ covering ∂P and a collection of maps $\{\varphi_j\}$ satisfying that for every k

1. *The extremal distance between $\partial\Omega_F \cap D_k$ and ∂U_k in U_k is uniformly bounded from above and bellow by $\frac{1}{M}$ and M respectively, where M is some numerical constant.*

2.
$$\omega(z_0, \ell_k; P) \lesssim \omega(z_0, D_k \cap \partial\Omega_F; \Omega_F) + \frac{1}{n}.$$

Proof. Let γ_+ denote the top part of ∂P, γ_- the horizontal symmetrization of γ_+ and include the disks intersecting the real line, denoted D_{j_L}, D_{j_R}, i.e., these two disks are included in both curves.

To define the dynamics we need to 'go one level down', i.e., to place inside each disk a rescaled and rotated copy of the disks covering γ_+ (or γ_- if the disk covers γ_-). Note that since every two consecutive disks the ratio between their radii is in the set $\left\{\frac{1}{2}, 1, 2\right\}$ then the line connecting their tangent points has length comparable with the radius of the disk. We denote by D_{jk} the copy of D_k inside D_j, and define the collection of domains and maps on the domains $\{U_{jk}\}$ where $U_{jk} = \frac{3}{2} D_{jk}$. Note that following the the first property of the polygon, P, every such domain only intersects three disks, D_{jk} and its neighbouring disks. To define $F_{jk} : U_{jk} \to \Omega$ we look at two cases- if $k \notin \{j_R, j_L\}$, then F_{jk} is rescaling and rotating mapping D_{jk} onto D_k and its neighbours to themselves. If $k \in \{j_L, j_R\}$, then we need to adjust our construction as these are our endpoints. We map the centre of the disk to the center of D_{j_L} (or D_{j_R}) and the two tangent points to the tangent points of D_{j_L} (or D_{j_R}) with its neighbours. The boundary is mapped to the boundary, because Mobiüs maps preserve angles.

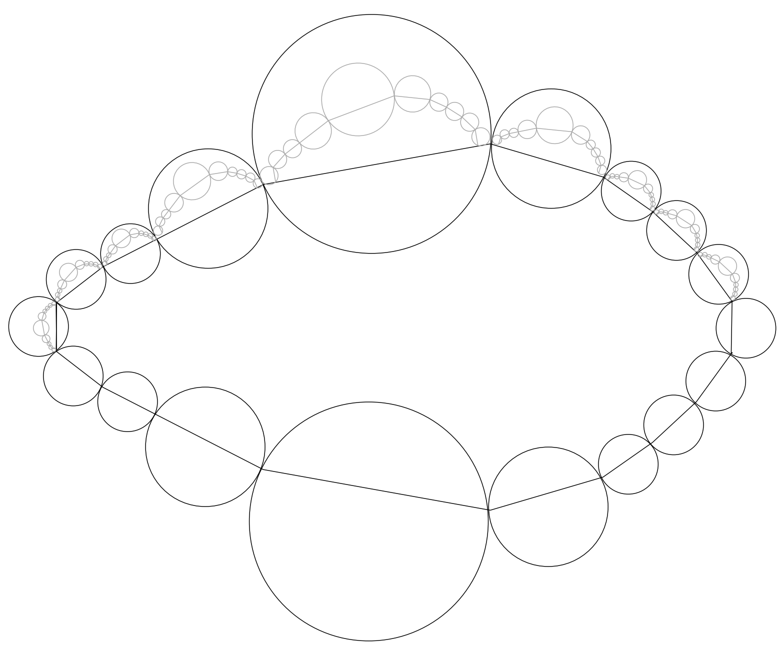


Figure 11

In addition, due to the first property of the polygon, P, combined with the fact that $diam(\partial P \cap D_j) \sim diam(D_j)$ implies that for every j the extremal distance between $\partial\Omega_F \cap D_j$ and ∂U_j in U_j is uniformly bounded from above satisfying the first property.

To see the second property holds let U denote the domain bounded by the collection of disks $\biguplus_{j,k} D_{jk}$, and let $P_0 := P \setminus \bigcup_{k=1}^m (D_{kj_L} \cup D_{kj_R})$. We will first show that $P_0 \subset U$, in other words, if you add to U each copy of the end-point disks, then we cover P. Note that, by definition, $\partial\Omega_F \subset \biguplus_{j,k} D_{jk}$ and therefore $P_0 \subseteq U \subseteq \Omega_F$.

Note that since P is symmetric and satisfy property 1, then the only disks intersecting the real line are the end-point disks. Let $\ell_j = \partial P \cap D_j$. Then when rescaling and rotating γ_+ to fit along ℓ_j, the disks D_{jk}, covering it, only intersects ℓ_j at the end-points. In other words, $P_0 = P \setminus \bigcup_{k=1}^m (D_{kj_L} \cup D_{kj_R}) \subset U$.

Now for every k,

$$\begin{aligned}\omega(z_0, \ell_k; P) &= \int_{\partial P_0} \omega(\zeta, \ell_k; P) d\omega(z_0, \zeta; P_0) = \omega(z_0, \ell_k \setminus (D_{kj_L} \cup D_{kj_R}); P_0) + \int_{\bigcup_{\nu=1}^m (\partial D_{\nu j_L} \cup \partial D_{\nu j_R})} \omega(\zeta, \ell_k; P) d\omega(z_0, \zeta; P_0) \\ &\leq \omega(z_0, \ell_k \setminus (D_{kj_L} \cup D_{kj_R}); P_0) + \int_{\bigcup_{\nu=1}^m (\partial D_{\nu j_L} \cup \partial D_{\nu j_R})} \frac{1}{n} d\omega(z_0, \zeta; P_0) \\ &\leq \omega(z_0, \ell_k \setminus (D_{kj_L} \cup D_{kj_R}); P_0) + \frac{1}{n},\end{aligned}$$

by Beurling's projection theorem.

A similar computation will give us:

$$\begin{aligned}\omega(z_0, D_k \cap \partial\Omega_F; \Omega_F) &\geq \omega(z_0, D_k \cap \partial\Omega_F; U \cup (\Omega_F \cap D_k)) \geq \int_{\partial P_0} \omega(\zeta, D_k \cap \partial\Omega_F; U \cup (\Omega_F \cap D_k)) d\omega(z_0, \zeta, P_0) \\ &\geq \int_{\ell_k \setminus (D_{kj_L} \cup D_{kj_R})} \omega(\zeta, D_k \cap \partial\Omega_F; U \cup (\Omega_F \cap D_k)) d\omega(z_0, \zeta, P_0) \\ &\geq \omega(z_0, \ell_k \setminus (D_{kj_L} \cup D_{kj_R}); P_0) \cdot \inf_{\zeta \in \ell_k \setminus (D_{kj_L} \cup D_{kj_R})} \omega(\zeta, D_k \cap \partial\Omega_F; U \cup (\Omega_F \cap D_k)) \\ &\geq c \cdot \omega(z_0, \ell_k \setminus (D_{kj_L} \cup D_{kj_R}); P_0),\end{aligned}$$

for some uniform $c \in (0,1)$ again, by Beurling's projection theorem. Note that c is uniform because we know that D_k does not contain any other part of the boundary. Overall, we see that

$$\omega(z_0, D_k \cap \partial\Omega_F; \Omega_F) + \frac{1}{n^2} \geq c \left(\omega(z_0, \ell_k \setminus (D_{kj_L} \cup D_{kj_R}); P_0) + \frac{1}{n} \right) \geq c \cdot \omega(z_0, \ell_k; P),$$

concluding our proof. □

Remark 5.12 *By placing rescaled copies of γ_- along parts of γ_+ and rescaled copies of γ_+ along parts of γ_- and using symmetric arguments as the ones above, one can create an iterated functions system with the second property being replaced by*

$$\omega(z_0, \ell_k; P) \gtrsim \omega(z_0, D_k \cap \partial\Omega_F; \Omega_F) - \frac{1}{n}.$$

The next lemma is a refinements of Carleson's estimate on the multiplicative constants of iterated functions systems. It is a quantified version that will allow up to propagate the 'good disks' used to define $N(\delta,\alpha,\eta)$ into smaller scales with a uniform error.

Definition 5.13 *We say an iterated functions system* **expands at rate at least** $D>1$ *if for every mapping in the system* $\inf_{U_j}|F_j'|\geq D$. *Equivalently, for every disk of radius* r, B, *in* U_j, *diam*$(F_j^{-1}(B))\leq \frac{r}{D}$.

Lemma 5.14 (Refined Carleson's estimate) *Let* F *be an iterated functions system expanding at rate at least* $D>1$, *and let* $Q_j := U_j\cap\Omega_F$, *where* $\{U_j\}$ *are the neighbourhoods where* F_j *are defined. Assume that the extremal distance between* ∂_j *and* $\partial Q_j\cap\Omega_F$ *in* Q_j *is uniformly bounded from above and bellow by* $\frac{1}{M}$ *and* M *respectively. Then there exists a constant* C *which depends only on* M, *so that*

$$\left|\frac{\omega\left(XYZ\right)}{\omega\left(XY\right)}\cdot\frac{\omega(Y)}{\omega\left(YZ\right)}-1\right|\leq C\cdot\left(\frac{1}{D}\right)^{|Y|-1}.$$

We relay on Makarov's proof in [44, p.52-53].

Proof. Let $g:\Omega_F\to R\mathbb{D}$ be a conformal map mapping z_0 to the origin for some R large. Without loss of generality we choose R large enough so that for every j, $\lambda_1(g(\partial_j))\geq 1$. For every j we denote by $G_j = g(Q_j)$, $\alpha_j = g(\partial_j)$, and $\sigma_j = g(\partial Q_j\cap\Omega_F)$. Let $h_j: G_j\to\mathbb{D}^+$ be a conformal map which maps σ_j to $\mathbb{T}^+$ and the centre of α_j to the origin. We then choose $\nu_j := g^{-1}\left(h_j^{-1}\left(\frac{i}{2}\right)\right)\in Q_j$, and denote by $\tilde{\alpha}_j = h_j(\alpha_j)$.

Let $X=(x_1,\cdots,x_n)$ be a cylinder in our system. We denote by $Q_X = F_{x_1}^{-1}\circ F_{x_2}^{-1}\circ\cdots\circ F_{x_{n-1}}^{-1}Q_{x_n}$, and by $\lambda_X = F_{x_1}^{-1}\circ F_{x_2}^{-1}\circ\cdots\circ F_{x_{n-1}}\lambda_{x_n}$. We will show that

$$\left|\frac{\omega(XYZ)}{\omega(XY)}\cdot\left(\frac{\omega\left(\nu_{Xy_1},\partial_{XYZ};Q_{Xy_1}\right)}{\omega\left(\nu_{Xy_1},\partial_{XY};Q_{Xy_1}\right)}\right)^{-1}-1\right|\lesssim\left(\frac{1}{D}\right)^{|Y|-1},$$

where the constant only depends on M. The same holds for $\frac{\omega(YZ)}{\omega(Y)}$ and $\frac{\omega\left(\nu_{y_1},\partial_{YZ};Q_{y_1}\right)}{\omega\left(\nu_{y_1},\partial_{Y};Q_{y_1}\right)}$ and as the second components are equal by conformal invariance of harmonic measures, this will conclude the proof.

Let $G=g(Q_{Xy_1})$ and define $h^X: G\to\mathbb{D}^+$ by $h^X(z) := h_{y_1}\left(g\circ F_X^{-1}\circ g^{-1}\right)$. Note that by definition,

$$h^X\circ g(\nu_{Xy_1}) = \frac{i}{2},\; h^X\circ g(\partial_{Xy_1}) = \tilde{\alpha}_{y_1},\text{ and } h^X\left(\sigma_{Xy_1}\right)=\mathbb{T}^+.$$

Denote by $\alpha := g(\partial_{Xy_1})$, $\beta := g(\partial_{XY})$, $\gamma := g(\partial_{XYZ})$ and by $\tilde{\alpha},\tilde{\beta}$, and $\tilde{\gamma}$ the images of α,β, and γ under h^X.

Since harmonic measure is conformal invariant,

$$\frac{\omega(XYZ)}{\omega(XY)} = \frac{\omega(0,\gamma;R\mathbb{D})}{\omega(0,\beta;R\mathbb{D})} = \frac{\lambda_1\left(\gamma\right)}{\lambda_1\left(\beta\right)},\quad\text{and}\quad \frac{\omega\left(\nu_{Xy_1},\partial_{XYZ};Q_{Xy_1}\right)}{\omega\left(\nu_{Xy_1},\partial_{XY};Q_{Xy_1}\right)} = \frac{\omega\left(\frac{i}{2},\tilde{\gamma};\mathbb{D}^+\right)}{\omega\left(\frac{i}{2},\tilde{\beta};\mathbb{D}^+\right)}.$$

We will first show that

$$\left|\frac{\lambda_1\left(\gamma\right)}{\lambda_1\left(\beta\right)}\left(\frac{\lambda_1\left(\tilde{\gamma}\right)}{\lambda_1\left(\tilde{\beta}\right)}\right)^{-1}-1\right|\lesssim\left(\frac{1}{D}\right)^{|Y|-1},$$

with a constant depending only on M.

Let $\hat{G}$ denote the symmetrization of G across α. Consider $\hat{h}^X$ as a map from the symmetrization, $\hat{h}^X : \hat{G} \to \mathbb{D}$. Because extremal distance is conformal invariant, and the extremal distance between ∂_{y_1} and $\partial Q_{y_1} \cap \Omega$ is assumed to be bounded between $\frac{1}{M}$ and M, the domain $\hat{G}$ and the compact set α satisfy the requirement of claim **??**. We conclude that

$$|h'(\zeta)| \asymp 1, \ \ |h''(\zeta)| \asymp 1, \ \ \forall \zeta \in \alpha,$$

and the constants depends on M alone. Then

$$\lambda_1\left(\tilde{\beta}\right) = \int\limits_{\beta} |h'(\zeta)|\, d|\zeta| \asymp \int\limits_{\beta} 1 d|\zeta| = \lambda_1(\beta) = \omega\left(\frac{i}{2}, \tilde{\beta}; \mathbb{D}^+\right) = \omega\left(\nu_{Xy_1}, \partial_{XY}; Q_{Xy_1}\right) = \omega\left(\nu_{y_1}, \partial_Y; Q_{y_1}\right).$$

Fix $\zeta_0 \in \gamma$, then

$$\left|\lambda_1\left(\tilde{\beta}\right) - \lambda_1(\beta)|h'(\zeta_0)|\right| = \left|\int\limits_{\beta} |h'(\zeta)|\, d|\zeta| - \lambda_1(\beta)|h'(\zeta_0)|\right| \leq \int\limits_{\beta} |h'(\zeta) - h'(\zeta_0)|\, d|\zeta| = \int\limits_{\beta} |h''(\xi_\zeta)|\,|\zeta - \zeta_0|\, d|\zeta| \lesssim \lambda_1(\beta)^2$$

$$\Rightarrow \left|\lambda_1(\beta)^{-1}\lambda_1\left(\tilde{\beta}\right) - |h'(\zeta_0)|\right| \leq C\lambda_1(\beta).$$

The same argument done with the curve γ shows that

$$\left|\lambda_1(\gamma)^{-1}\lambda_1(\tilde{\gamma}) - |h'(\zeta_0)|\right| \leq C\lambda_1(\gamma).$$

This implies that

$$\left|\frac{\lambda_1(\gamma)}{\lambda_1(\beta)}\left(\frac{\lambda_1(\tilde{\gamma})}{\lambda_1\left(\tilde{\beta}\right)}\right)^{-1} - 1\right| = \left|\frac{\lambda_1\left(\tilde{\beta}\right)\lambda_1(\beta)^{-1}}{\lambda_1(\tilde{\gamma})\lambda_1(\gamma)^{-1}} - 1\right| \leq \left|\frac{|h'(\zeta_0)| + C\lambda_1(\beta)}{|h'(\zeta_0)| - C\lambda_1(\gamma)} - 1\right| = \frac{C\left(\lambda_1(\beta) + \lambda_1(\gamma)\right)}{|h'(\zeta_0)| - C\lambda_1(\gamma)} \lesssim \lambda_1(\beta),$$

where the constant depends only on M. To conclude this part of the proof it is left to show that $\lambda_1(\beta) \lesssim \left(\frac{1}{D}\right)^{|Y|-1}$ with a constant depending only on M.

We note that since the extremal distance between $\frac{i}{2}$ and $\partial\mathbb{D}^+$ is some constant (for definition see [29, p.144 bottom]), then extremal distance between ∂Q_{y_1} and ν_{y_1} is uniformly bounded from above and bellow by some uniform constants as a conformal image of the half disk. This implies that for every arc in the boundary of ∂Q_{y_1}, $\omega\left(\nu_{y_1}, A; Q_{y_1}\right) \lesssim \frac{\lambda_1(A)}{diam(Q_{y_1})}$, where the constant is a numerical constant. Overall

$$\lambda_1(\beta) \sim \omega\left(\frac{i}{2}, \tilde{\beta}; \mathbb{D}^+\right) = \omega\left(\nu_{Xy_1}, \partial_{XY}; Q_{Xy_1}\right) = \omega\left(\nu_{y_1}, \partial_Y; Q_{y_1}\right) \sim \frac{\lambda_1(\partial_Y)}{diam(Q_{y_1})}$$
$$= \frac{\lambda_1\left(F_{y_n}^{-1} \circ \cdots \circ F_{y_2}^{-1}\partial_{y_1}\right)}{diam(Q_{y_1})} \lesssim \left(\frac{1}{D}\right)^{|Y|-1},$$

where the constant depends on M alone.

To conclude the proof, it is left to show that

$$\left|\frac{\lambda_1(\tilde{\gamma})}{\lambda_1\left(\tilde{\beta}\right)} : \frac{\omega\left(\frac{i}{2}, \tilde{\gamma}; \mathbb{D}^+\right)}{\omega\left(\frac{i}{2}, \tilde{\beta}; \mathbb{D}^+\right)} - 1\right| \lesssim \left(\frac{1}{D}\right)^{|Y|-1}.$$

Let $\phi : \mathbb{D}^+ \to \mathbb{D}$ be a conformal map, mapping $\frac{i}{2}$ to the origin, and $\mathbb{T}^+$ to itself. Then for every arc $A \subset \partial\mathbb{D}^+$,

$$\omega\left(\frac{i}{2}, A; \mathbb{D}^+\right) = \omega(0, \phi(A); \mathbb{D}) = \lambda_1(\phi(A)).$$

Now, ϕ is a fixed Möbius map, and therefore its second derivative is uniformly bounded as a function of the distance of $\tilde{\alpha}$ from ± 1, which is equal (up to a uniform constant) to the extremal distance between ∂_{y_1} and $\partial Q_{y_1} \cap \Omega$, in Q_{y_1}, in other words, it depends on M. Fix $\tilde{\zeta}_0 \in \tilde{\gamma}$, then , as before, for every arc A containing $\tilde{\gamma}$,

$$\begin{aligned}\left|\lambda_1(\phi(A)) - \lambda_1(A)\left|\phi'(\tilde{\zeta}_0)\right|\right| &\le \int_A \left||\phi'(\zeta)| - \left|\phi'(\tilde{\zeta}_0)\right|\right| d|\zeta| \le \int_A \left|\phi'(\zeta) - \phi'(\tilde{\zeta}_0)\right| d|\zeta| \\ &= \int_A |\phi''(\xi_\zeta)| \left|\zeta - \tilde{\zeta}_0\right| d|\zeta| \lesssim \lambda_1(A)^2,\end{aligned}$$

where the constant only depends on the the bounds we had for the second derivative, which in turn depends on M. Overall,

$$\begin{aligned}\left|\frac{\omega\left(\frac{i}{2}, \tilde{\gamma}; \mathbb{D}^+\right)}{\omega\left(\frac{i}{2}, \tilde{\beta}; \mathbb{D}^+\right)} \cdot \left(\frac{\lambda_1(\tilde{\gamma})}{\lambda_1\left(\tilde{\beta}\right)}\right)^{-1} - 1\right| = \left|\frac{\lambda_1(\phi(\tilde{\gamma}))}{\lambda_1(\phi(\tilde{\beta}))} \cdot \left(\frac{\lambda_1(\tilde{\gamma})}{\lambda_1\left(\tilde{\beta}\right)}\right)^{-1} - 1\right| &\le \frac{\left(\lambda_1(\tilde{\gamma})\left|\phi'(\tilde{\zeta}_0)\right| + \lambda_1(\tilde{\gamma})^2\right)\lambda_1(\tilde{\gamma})^{-1}}{\left(\lambda_1(\tilde{\beta})\left|\phi'(\tilde{\zeta}_0)\right| + \lambda_1(\tilde{\beta})^2\right)\lambda_1\left(\tilde{\beta}\right)^{-1}} - 1 \\ &\le \frac{\left|\phi''(\tilde{\zeta}_0)\right|^{-1}\left(\lambda_1(\tilde{\gamma}) + \lambda_1(\tilde{\beta})\right)}{1 + \left|\phi''(\tilde{\zeta}_0)\right|^{-1}\lambda_1(\tilde{\beta})} \lesssim \lambda_1(\tilde{\beta}) \lesssim \lambda_1(\beta) \lesssim \left(\frac{1}{D}\right)^{|Y|-1},\end{aligned}$$

and the constants, as before, only depend on M. $\square$

5.3.1 Minkowski dimension spectrum and Minkowski distortion spectrum of words

One way to describe an iterated functions system is to use symbolic dynamics. Let $\varphi : W \to \partial\Omega$ be the map taking infinite words (from the set W) into their corresponding points in $\partial\Omega$. We shall abuse the notation of F to denote the domain generated by the system F. For a cylinder set $[a_1, \cdots, a_k]$ we interpret $\varphi[a_1, \cdots, a_k] := \left\{\zeta \in \partial\Omega, \varphi^{-1}(\zeta) \in [a_1, \cdots, a_k]\right\}$ or in other words, it is the collection of points in $\partial\Omega$ whose 'word' description begins with the letters $a_1, \cdots, a_k$. Let $d := \min_{a\in\Sigma} diam(\varphi[a]),\ D := \max_{a\in\Sigma} diam(\varphi[a])$.

We would like to present similar definitions for the dimension and the distortion spectrums in the context of words.

5.3.1.1 Definitions:

Let $\Sigma = \{a_1, \cdots, a_N\}$ denote the finite alphabet used in the symbolic dynamics description of our iterated functions system. We abuse the notation of diameter and harmonic measure of words by defining $diam(a_j) = diam(\varphi[a_j])$ and $\omega(a_j) = \omega(z_0, \varphi([a_j]); F)$, and denote by $|w|$ the length of the word w.

Given $m \in \mathbb{N}$ we denote by

$$I^{[m]} := \left\{(k_1, \cdots, k_N); \sum_{j=1}^{N} k_j = m, k_j \in \mathbb{N} \cup \{0\}\right\}.$$

Given a sequence $(k_1, \cdots, k_N) \in I^{[m]}$ we say $w = (w_1, \cdots, w_m) \in W^{(k_1, \cdots, k_N)}$ if for every $1 \le j \le N$

$$\# \{\nu, w_\nu = a_j\} = k_j.$$

Definition 5.15 *We define the* **Minkowski word upper dimension spectrum** *by*

$$f_\Omega^{+word}(\alpha) = \lim_{\eta \to 0} \limsup_{m \to \infty} \sup_{(k_1, \cdots, k_N) \in I^{[m]}} \frac{\log N^+_{word}((k_1, \cdots, k_N), \alpha, \eta)}{\sum_{j=1}^{N} k_j \log(diam(a_j))},$$

where $N^+_{word}((k_1, \cdots, k_N), \alpha, \eta)$ *is the maximal number of disjoint words* $w \in W^{(k_1, \cdots, k_N)}$ *satisfying that*

$$\omega(z_0, \varphi[w]; F) \ge \prod_{j=1}^{N} diam(a_j)^{k_j(\alpha+\eta)}.$$

Similarly, we define the **Minkowski word lower dimension spectrum** *by*

$$f_\Omega^{-word}(\alpha) = \lim_{\eta \to 0} \limsup_{m \to \infty} \sup_{(k_1, \cdots, k_N) \in I^{[m]}} \frac{\log N^-_{word}((k_1, \cdots, k_N), \alpha, \eta)}{\sum_{j=1}^{N} k_j \log(diam(a_j))},$$

where $N^-_{word}((k_1, \cdots, k_N), \alpha, \eta)$ *is the maximal number of disjoint words* $w \in W^{(k_1, \cdots, k_N)}$ *satisfying that*

$$\omega(z_0, \varphi[w]; F) \le \prod_{j=1}^{N} diam(a_j)^{k_j(\alpha-\eta)}$$

Note that by using d^{curve} instead of d, there is no reason to define an equivalent d^{word} as it is just the same. For our convenience we shall define the collection of curves

$$\Gamma(a, (k_1, \cdots, k_n)) = \Gamma\left(a, 1 - \prod_{j=1}^{n} diam(a_j)^{\frac{k_j}{1-a}}\right).$$

5.3.1.2 Consistency:

Lemma 5.16 *Let F be a finite iterated functions system. Then*

1. $f_F^{+word}(\alpha) = f_F^+(\alpha)$.

2. $f_F^{-word}(\alpha) = f_F^-(\alpha)$.

Proof. We will first show that there is a one to one correspondence between good disks and good words. Because the harmonic measure of a finite iterated functions system is doubling, it is clear that if a word is good, then there exists a disk of double the diameter which is good with bounded multiplicative error. To see the reverse correspondence, fix $r > 0$ and let B be a good disk of diameter r. We will show that for every η and every r small enough (depending on η and the domain) there exists a finite word (or a cylinder) $[w_0]_0^k$ so that

1. $r^{1+\eta} \le diam(\varphi([w_0]_0^k)) \le r$.

2. $\omega(B)^{1+2\eta} \leq \omega(\varphi([w_0]_0^k)) \leq \omega(B)^{1-\eta}$.

Because the harmonic measure of a finite iterated functions system is doubling, we may assume without loss of generality that there exists an infinite word so that $\varphi(w_0)$ is the centre of the disk, B (otherwise because this correspondence is defined almost surely, we can shift the disk slightly and change the harmonic measure by at most a constant). Let

$$k_{min} := \min \left\{ \nu \in \mathbb{N}, diam(\varphi([w_0]_0^\nu)) \leq r \right\}.$$

On one hand,

$$r \geq diam(\varphi([w_0]_0^k)) \geq diam(\varphi([w_0]_0^{k-1})) \cdot \min_{1\leq j\leq N} diam(a_j) > r \cdot d = r^{1+\eta}$$

as long as r is small enough (depending on η and d). On the other hand, because $w_0 \in D \cap \varphi([w_0]_0^k)$, then

$$2D = B(\varphi(w_0), 2r) \supseteq B(\varphi(w_0), 2diam(\varphi([w_0]_0^{k-1}))) \supseteq \varphi([w_0]_0^k).$$

Because the harmonic measure is doubling this implies that

$$\omega(\varphi([w_0]_0^k)) \leq \omega(2B) \sim \omega(B).$$

Similarly,

$$\varphi([w_0]_0^k) \supseteq B\left(\varphi(w_0), \frac{1}{2} \cdot diam(\varphi([w_0]_0^{k-1}))\right) \supseteq B\left(\varphi(w_0), \frac{1}{2} r^{1+\eta}\right),$$

which implies that

$$\omega(\varphi([w_0]_0^k)) \gtrsim \omega(B)^{1+\eta},$$

Now, it is left to note that every sequence $\delta_\nu \searrow 0$ corresponds to a sequence of configurations $(k_1^\nu, \cdots k_N^\nu) \in I^{[m_\nu]}$ satisfying 1 and 2 and vise-verse concluding the proof. □

5.3.1.3 Propagation:

Definition 5.17 *Let $\Omega \subset \mathbb{C}$ be a domain. We say Ω* **propagates the function** *$\varphi(\delta, \alpha, \eta)$ if there exists a constant C, so that for every δ small enough (which may depend on η, α and the domain, Ω), and for every $n \in \mathbb{N}$*

$$\frac{\log \varphi\left(\delta^{2^n}, \alpha, C \cdot \eta\right)}{\log\left(\frac{1}{\delta^{2^n}}\right)} \geq \frac{\log \varphi(\delta, \alpha, \eta)}{\log\left(\frac{1}{\delta}\right)}.$$

The first observation is that for iterated functions systems, the functions $N^{\pm word}$ and $\#\Gamma$ propagate, making this property interesting.

Observation 5.18 *Let F be a finite iterated functions system. Fix $m \in \mathbb{N}$ and $(k_1, \cdots, k_N) \in I^{[m]}$, and define $\delta := \prod_{j=1}^{N} diam(a_j)^{n_j}$.*

1. *The function $N^{\pm word}((k_1, \cdots, k_N), \alpha, \eta)$ propagates.*

2. *The function $\#\Gamma(a,(k_1,\cdots,k_N))$ propagates.*

In both cases the constant C is a constant that will depend on the constant from Carleson's Lemma, Lemma 5.14.

Proof. We will show the proof for f^{+words} the other two cases are identical.

For every ν we define the configuration $(\nu\cdot k_1,\nu\cdot k_2,\cdots,\nu\cdot k_N)\in I^{[\nu\cdot m]}$. Let $w_1,w_2,\cdots,w_\nu\in W^{(k_1,\cdots,k_N)}$ be so that for every ℓ

$$\omega(z_0,\varphi[w_\ell];F)\geq\prod_{j=1}^{N} diam(a_j)^{k_j(\alpha+\eta)}.$$

For every $\nu\in\mathbb{N}$ for every word $w=w_1w_2\cdots w_\nu$ we have

$$\begin{aligned}\omega(z_0,\varphi[w];F)\geq A^{-\nu}\prod_{\ell=1}^{\nu}\omega(z_0,\varphi[w_\ell];F)>A^{-\nu}\prod_{\ell=1}^{\nu}\prod_{j=1}^{N} diam(a_j)^{k_j(\alpha+\eta)}=A^{-\nu}\prod_{j=1}^{N} diam(a_j)^{k_j\cdot\nu(\alpha+\eta)}\\ \geq\prod_{j=1}^{N} diam(a_j)^{k_j\cdot\nu\left(\alpha+\eta+\frac{D}{m}\right)}>\prod_{j=1}^{N} diam(a_j)^{k_j\cdot\nu(\alpha+C\eta)}\end{aligned}$$

where A is the constant from Lemma 5.14, D and C are constants depending on A, and the last inequality holds for all m large enough (depending on η and A). □

The next lemma shows that because the functions N^{+word} and $\#\Gamma$ propagate with constants that depend on the modulus $\partial_j:=\partial F\cap D_j$ and $\partial\left(\frac{3}{2}D_j\cap\Omega_F\right)$ in $\frac{3}{2}D_j$ then (1) and (2) hold, concluding the proof of Theorem 2.4.

Lemma 5.19 *(1) holds.* *(2) holds.*

Proof. **(1) holds:** Fix $\varepsilon>0$ and let Ω be a domain satisfying $f^+_\Omega(\alpha)>F(\alpha)-\varepsilon$. There exists $\eta_0>0$ and $\delta_0>0$ small enough satisfying that $\log\left(N^+_\Omega(\delta,\alpha,\eta)\right)>(F(\alpha)-\varepsilon)\log\left(\frac{1}{\delta}\right)$. Let P be the polygon constructed in Theorem 5.9 and let F be the iterated functions system constructed from P in Lemma 5.11. Then,

$$N^+_P(\delta_0,\alpha,\eta_0\,(5\alpha+12))\gtrsim N^+_\Omega(\delta_0,\alpha,\eta_0),$$

and Observation 5.18 implies that for every $\nu\in\mathbb{N}$ and every $\eta>0$

$$\begin{aligned}N^+_F(\delta_0^\nu,\alpha+C\cdot\eta_0\,(5\alpha+12)\,,\eta)&=N^+_F(\delta_0^\nu,\alpha+\eta,C\cdot\eta_0\,(5\alpha+12))\\&\geq\left(N^+_F\left(\delta_0,\alpha+\eta,\eta_0\,(5\alpha+12)\right)\right)^\nu\geq\left(A\cdot N^+_\Omega(\delta_0,\alpha,\eta_0)\right)^\nu.\end{aligned}$$

Then for every η_0 fixed

$$\begin{aligned}f^+_F(\alpha+C\cdot\eta_0\,(5\alpha+12))&=\lim_{\eta\to0}\limsup_{\delta\to0}\frac{\log\left(N^+_F(\delta,\alpha+C\cdot\eta_0\,(5\alpha+12)\,,\eta)\right)}{\log\left(\frac{1}{\delta}\right)}\\&\geq\lim_{\eta\to0}\limsup_{\nu\to\infty}\frac{\log\left(N^+_F(\delta_0^\nu,\alpha+C\cdot\eta_0\,(5\alpha+12)\,,\eta)\right)}{\nu\log\left(\frac{1}{\delta_0}\right)}\geq\lim_{\eta\to0}\limsup_{\nu\to\infty}\frac{\log\left(\left(A\cdot N^+_\Omega(\delta_0,\alpha,\eta_0)\right)^\nu\right)}{\nu\log\left(\frac{1}{\delta_0}\right)}\\&=\lim_{\eta\to0}\limsup_{\nu\to\infty}\frac{\log\left(N^+_\Omega(\delta_0,\alpha,\eta_0)\right)}{\log\left(\frac{1}{\delta_0}\right)}+\frac{\log(A)}{\log\left(\frac{1}{\delta_0}\right)}\geq F(\alpha)-2\varepsilon,\end{aligned}$$

assuming δ_0 was numerically small enough. To conclude the proof we note that f^+ is upper semi-continuous. This combined with the fact that C is a uniform constant, gives that

$$\sup_{F\text{ IFS}} f_F^+(\alpha) \geq \sup_{F\text{ IFS}} \lim_{\alpha'\searrow\alpha} f_F^+(\alpha') \geq \sup_{F\text{ IFS}} \lim_{\eta_0\to 0} f_F^+(\alpha + C\cdot\eta_0\,(5\alpha+12)) \geq F(\alpha) - 2\varepsilon.$$

(2) holds: Fix $a>0$, $\varepsilon>0$ and let Ω be a domain satisfying $d_\Omega(a) > D(a)-\varepsilon$. There exists $a'>a$ close enough and r close enough to 1 satisfying that

$$d_\Omega(a) \leq \frac{\log\left(\#\Gamma\left(a',r\right)\right)}{\log\left(\frac{1}{1-r}\right)} + \varepsilon$$

Let P be the polygon constructed in Theorem 5.10 and let F be the iterated functions system constructed from P in Lemma 5.11. Then, $\#\Gamma_P(a'',r') \gtrsim \#\Gamma_{\Omega_0}(a',r)$ with $a'' = a\left(1+O(|a-a'|)\right)$, $1-r' = (1-r)^{1+5\left|a-a'\right|}$, and following Observation 5.18

$$d_F^{curves}(a'') = \frac{\log\left(\#\Gamma_P(a'',r')\right)}{\log\left(\frac{1}{1-r'}\right)} \geq \frac{\log\left(\#\Gamma_{\Omega_0}(a',r)\right)}{\log\left(\frac{1}{1-r}\right)}\left(1-O(|a-a'|)\right).$$

Since this is true with uniform constants, we see that for every $\varepsilon>0$ and for every $a'>a$,

$$\sup_{F\text{ IFS}} d_F(a) = \lim_{a''\to a}\sup_{F\text{ IFS}} d_F(a'') = \sup_{F\text{ IFS}} d_F^{curve}(a'') \geq \frac{\log\left(\#\Gamma_{\Omega_0}(a',r)\right)}{\log\left(\frac{1}{1-r}\right)}\left(1-O(|a-a'|)\right) \geq D(a) - 2\varepsilon,$$

as d is upper semi-continuous for $a>0$, concluding the proof. □

6 References

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